Warmup:
Finish up Lecture 13 Exercises
(See end of packet.)

Selected answers
and hints also on Moodle

Lecture 14:
Row operations as matrices
Inverses

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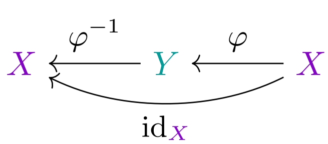
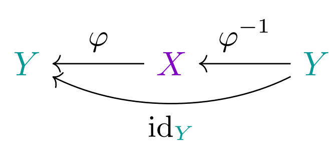
Homework 6 extension coming...
BUT tonight is still the last day to get help.

Recall: A function $f : X \to Y$ is invertible if and only if it's bijective.
What does that mean in terms of matrices?

Let X and Y be finite-dimensional vector spaces over a field F, and let $\varphi : X \to Y$ be a bijective linear function, a.k.a. an **isomorphism**. Since φ is bijective, there exists a (two-sided) inverse $\varphi^{-1} : Y \to X$ such that

$$\varphi^{-1} \circ \varphi = \mathrm{id}_X \qquad \text{and} \qquad \varphi \circ \varphi^{-1} = \mathrm{id}_Y .$$





Q. Given an isomorphism $\varphi : X \to Y$, how can I compute φ^{-1}?
Q. If φ is defined by a matrix A (i.e. defined on an ordered basis),
what is the matrix A^{-1} associated to φ^{-1}? (A^{-1} is called the **inverse** of A.)

Some things we know:

Rank-nullity says $\dim(X) = \operatorname{rank}(\varphi) + \operatorname{nullity}(\varphi)$.

But φ injective implies $\operatorname{nullity}(\varphi) = 0$; and
φ surjective implies that $\operatorname{rank}(\varphi) = \dim(Y)$.

So

$$\boxed{\dim(X) = \dim(Y) = \operatorname{rank}(\varphi)}$$

"Isomorphisms preserve dimension."

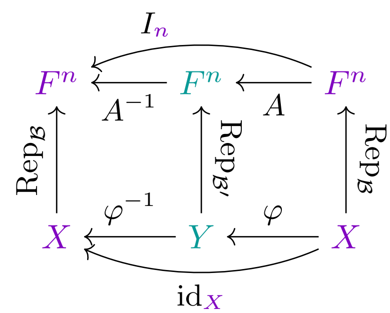
"Isomorphisms have full rank."

In terms of the associated matrices:

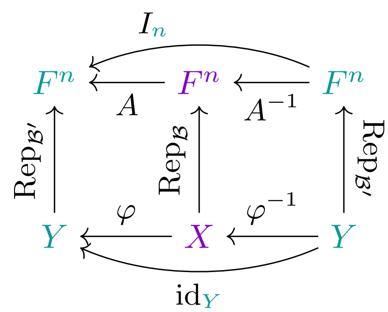
If $A = \operatorname{Rep}_{\mathcal{B}}^{\mathcal{B}'}(\varphi)$ (with respect to fixed ordered bases $\mathcal{B} \subseteq X$ and $\mathcal{B}' \subseteq Y$), then

1. $A \in M_n(F)$, where $n = \dim(X) = \dim(Y)$; and
2. $\operatorname{rank}(A) = n$. "$A$ is **nonsingular**"

$$\boxed{A^{-1}A = I_n.} \qquad \boxed{AA^{-1} = I_n.}$$



and



Conversely, suppose $A \in M_n(F)$ encodes a function $\varphi : X \to Y$ (i.e. $\dim(X)$ and $\dim(Y)$ are both n, and $A = \operatorname{Rep}_{\mathcal{B}}^{\mathcal{B}'}(\varphi)$ w.r.t. some ordered bases $\mathcal{B} \subseteq X$ and $\mathcal{B}' \subseteq Y$).

If, additionally, we know that $\operatorname{rank}(A) = n\ldots$

- First, $n = \operatorname{rank}(A) = \operatorname{rank}(\varphi) = \dim(Y)$. So since Y is finite-dimensional and $\mathcal{R}(\varphi) \subseteq Y$ is a subspace, we know $\mathcal{R}(\varphi) = Y$. So φ is surjective.
- Rank-nullity says

$$n = \overbrace{\dim(X) = \operatorname{rank}(\varphi) + \operatorname{nullity}(\varphi)}^{\text{Rank-nullity Theorem}} = n + \operatorname{nullity}(\varphi).$$

  So $\operatorname{nullity}(\varphi) = 0$; and hence φ is also injective.

So φ is an isomorphism.

Theorem. Let X and Y be finite-dimensional vector spaces over F, and let $\varphi : X \to Y$ be a linear function.

Then φ is an isomorphism if any only if

$$\underbrace{\text{any associated matrix } A \text{ is square and has full rank}}_{(*)}.$$

$(*)$ More precisely:
For any ordered bases $\mathcal{B} \subseteq X$ and $\mathcal{B}' \subseteq Y$, we have

(1) $|\mathcal{B}| = |\mathcal{B}'|$;

(2) $A = \mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}'}(\varphi) \in M_n(F)$, where $n = |\mathcal{B}|$; and

(3) $\mathrm{rank}(A) = n$. "A is **nonsingular**"

Proof. See above; use Rank-Nullity theorem.

[CAUTION: This whole theorem makes less sense (and is false) if X is not finite-dimensional. See Homework 6.]

Corollary. A linear function φ associated to a matrix $A \in M_{k,\ell}(F)$ is an isomorphism if and only if A row reduces to an identity matrix.

Example. At the end of Lecture 12, we computed that with respect to the standard ordered bases

$$\mathcal{A} = \langle 1, x, x^2, x^3 \rangle \subseteq \mathcal{P}_3(\mathbb{R}) \quad \text{and} \quad \mathcal{B} = \langle E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2} \rangle \subseteq M_2(\mathbb{R}),$$

the function

$$f : \mathcal{P}_3(\mathbb{R}) \to M_2(\mathbb{R}) \quad \text{defined by} \quad f(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}.$$

has matrix representation

$$A = \mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}.$$

Now, A row reduces as

$$A \overset{\cdots}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 9 & 27 \end{pmatrix} \overset{\cdots}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 6 & 24 \end{pmatrix} \overset{\cdots}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\cdots}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4.$$

So f is an isomorphism. But what *is* its inverse???

Row reduction operators

Goal: Use the row reduction algorithm to compute A^{-1}/φ^{-1}.

Recall that the three row operations were

$$\mathbf{row}_i \leftrightarrow \mathbf{row}_i, \quad \mathbf{row}_i \mapsto \lambda\mathbf{row}_i, \quad \mathbf{row}_i \mapsto \mathbf{row}_i + \mu\mathbf{row}_j,$$

where $\lambda, \mu \in F$ with $\lambda \neq 0$.

Key insight: On the Lecture 13 exercises, you saw that $E_{i,j}X$ takes the jth row of X and inserts it into the ith row of a matrix that's elsewhere all 0. For example, in $M_3(F)$,

$$E_{1,2}X = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{pmatrix} = \begin{pmatrix} X_{2,1} & X_{2,2} & X_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we can build row operations by taking the right linear combinations of $E_{i,j}$'s acting on the left!

1. Row swapping. To swap $\mathbf{row}_i$ and $\mathbf{row}_j$, add up:

- insert $\mathbf{row}_i$ into row j; $E_{j,i}$
- insert $\mathbf{row}_j$ into row i; $E_{i,j}$
- insert $\mathbf{row}_k$ into into row k for all $k \neq i, j$. $E_{k,k}$

Define
$$P_{i,j} = E_{i,j} + E_{j,i} + \sum_{\substack{k=1,\ldots,n \\ k \neq i,j}} E_{k,k}.$$

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2. Row scaling. To scale $\mathbf{row}_i$ by λ, add up:

- insert $\mathbf{row}_i$ into row i and scale by λ; $\lambda E_{i,i}$
- insert $\mathbf{row}_k$ into into row k for all $k \neq i$. $E_{k,k}$

Define
$$S_i(\lambda) = \lambda E_{i,i} + \sum_{\substack{k=1,\ldots,n \\ k \neq i}} E_{k,k}.$$
[Book: $M_i(\lambda)$]

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3. Row Combination. To replace $\mathbf{row}_i$ by $\mathbf{row}_i + \lambda\mathbf{row}_j$, add up:

- insert $\mathbf{row}_j$ into row i and scale by λ; $\lambda E_{i,j}$
- insert $\mathbf{row}_i$ into row i; $E_{i,i}$
- insert $\mathbf{row}_k$ into into row k for all $k \neq i$. $E_{k,k}$

Define
$$C_{j,i}(\lambda) = \lambda E_{i,j} + I_n.$$

For example, in $M_4(F)$,

$$
\begin{aligned}
P_{1,3}X &= (E_{1,3} + E_{3,1} + E_{2,2} + E_{4,4})X \\
&= E_{1,3}X + E_{3,1}X + E_{2,2}X + E_{4,4}X \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} \\
&= \begin{pmatrix} X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} \\
&= \begin{pmatrix} X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix},
\end{aligned}
$$

which is exactly the result of swapping $\mathbf{row}_1 \leftrightarrow \mathbf{row}_3$!

For example, in $M_4(F)$,

$$
\begin{aligned}
S_1(\lambda)X &= (\lambda E_{1,1} + E_{2,2} + E_{3,3} + E_{4,4})X \\
&= \lambda E_{1,1}X + E_{2,2}X + E_{3,3}X + E_{4,4}X \\
&= \lambda\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} \\
&= \begin{pmatrix} \lambda X_{1,1} & \lambda X_{1,2} & \lambda X_{1,3} & \lambda X_{1,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix} \\
&= \begin{pmatrix} \lambda X_{1,1} & \lambda X_{1,2} & \lambda X_{1,3} & \lambda X_{1,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} \end{pmatrix},
\end{aligned}
$$

which is exactly the result of scaling $\mathbf{row}_1$ by λ!

Permute $\mathbf{row}_i \leftrightarrow \mathbf{row}_j$: $P_{i,j} = E_{i,j} + E_{j,i} + \sum_{\substack{k=1,\dots,n \\ k\neq i,j}} E_{k,k}$

Scale $\mathbf{row}_i \mapsto \lambda\mathbf{row}_i$: $S_i(\lambda) = \lambda E_{i,i} + \sum_{\substack{k=1,\dots,n \\ k\neq i}} E_{k,k}$

Combine $\mathbf{row}_j \mapsto \lambda\mathbf{row}_i + \mathbf{row}_j$: $C_{i,j}(\lambda) = \lambda E_{j,i} + I_n$

Back to our example from Lecture 12:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \xmapsto{\mathbf{row}_2 \mapsto \mathbf{row}_2 - \mathbf{row}_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} = C_{1,2}(-1)A$$

$$\xmapsto{\mathbf{row}_3 \mapsto \mathbf{row}_3 - \mathbf{row}_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} = C_{1,3}(-1)(C_{1,2}(-1)A)$$

$$\xmapsto{\mathbf{row}_4 \mapsto \mathbf{row}_4 - \mathbf{row}_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 9 & 27 \end{pmatrix}$$

$$= C_{1,4}(-1)(C_{1,3}(-1)(C_{1,2}(-1)A))$$

$$\vdots$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \xmapsto{C_{1,2}(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \xmapsto{C_{1,3}(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}$$

$$\xmapsto{C_{1,4}(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 9 & 27 \end{pmatrix} \xmapsto{C_{2,3}(-2)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 3 & 9 & 27 \end{pmatrix} \xmapsto{C_{2,4}(-3)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 6 & 24 \end{pmatrix}$$

$$\xmapsto{S_3(1/2)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 6 & 24 \end{pmatrix} \xmapsto{S_4(1/6)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xmapsto{C_{3,2}(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xmapsto{C_{3,4}(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xmapsto{C_{4,2}(2)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xmapsto{C_{4,3}(-3)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{C_{4,3}(\text{-}3)C_{4,2}(2)C_{3,4}(\text{-}1)C_{3,2}(\text{-}1)S_4(\tfrac{1}{6})S_3(\tfrac{1}{2})C_{2,4}(\text{-}3)C_{2,3}(\text{-}2)C_{1,4}(\text{-}1)C_{1,3}(\text{-}1)C_{1,2}(\text{-}1)}_{A^{-1}} A = I_4$$

To check, compute

$C_{4,3}(\text{-}3)C_{4,2}(2)C_{3,4}(\text{-}1)C_{3,2}(\text{-}1)S_4(1/6)S_3(1/2)C_{2,4}(\text{-}3)C_{2,3}(\text{-}2)C_{1,4}(\text{-}1)C_{1,3}(\text{-}1)C_{1,2}(\text{-}1)$

$$= \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{-3} \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{2} \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-1} & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1/6} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1/2} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{-3} & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{-2} & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ \mathbf{-1} & 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ \mathbf{-1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ \mathbf{-1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \text{-}11/6 & 3 & \text{-}3/2 & 1/3 \\ 1 & \text{-}5/2 & 2 & \text{-}1/2 \\ \text{-}1/6 & 1/2 & \text{-}1/2 & 1/6 \end{pmatrix},$$ and verify

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \text{-}11/6 & 3 & \text{-}3/2 & 1/3 \\ 1 & \text{-}5/2 & 2 & \text{-}1/2 \\ \text{-}1/6 & 1/2 & \text{-}1/2 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

You try:

1. Row reduce $A = \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}$ **ONE** step at a time, and use that process to build the inverse of A. *Actually compute* the result, and then multiply it by A to check your answer.

2. Repeat #1 for $A = \begin{pmatrix} 0 & 1 & 5 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$.

3. What is the inverse of $P_{i,j}$? of $S_k(\lambda)$? of $C_{\ell,m}(\lambda)$?

[Hint: How do you undo each corresponding row operation?]

Lecture 13 exercises

1. Let
$$A = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & -1 \end{pmatrix}.$$

   (a) For each of the following, decide whether or not the product is defined. If so, compute it; if not, say why not. **(i)** AB, **(ii)** BA, **(iii)** AC, **(iv)** CA, **(v)** BC, **(vi)** CB, **(vii)** CD, **(viii)** DC.

   (b) Compare $(CA)B$ (multiply CA and B) and $C(AB)$ (multiply C and AB).

   (c) For $\ell \in \mathbb{Z}_{\geq 1}$, we denote $A^\ell := \overbrace{AA \cdots A}^{\ell \text{ terms}}$. For each of the following, decide whether or not the product is defined. If so, compute it; if not, say why not. **(i)** A^2 **(ii)** B^2 **(iii)** C^2 **(iv)** D^2

2. Recall $E_{i,j}$ denotes a matrix with a 1 in row i, col j and 0's elsewhere.

   (a) Working in $M_3(F) = M_{3,3}(F)$, let
$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{pmatrix}.$$
   Compute **(i)** $E_{1,2}X$, **(ii)** $XE_{1,2}$, **(iii)** $E_{3,3}X$, **(iv)** $XE_{3,3}$.

   (b) Working more generally over $M_n(F) = M_{n,n}(F)$, let $X \in M_n(F)$ and let $1 \leq i, j, k, \ell \leq n$. Describe/conjecture the following.[1] **(i)** $E_{i,j}X$, **(ii)** $XE_{i,j}$, **(iii)** $E_{i,j}XE_{k,\ell}$, **(iv)** $E_{i,j}E_{k,\ell}$.

3. The **identity matrix** I_n is the $n \times n$ matrix with 1 and (i,i)-entry for $i = 1, \ldots, n$, and 0's elsewhere. For example,
$$I_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) If $f : F^3 \to F^3$ is the function associated to I_3, compute $f((x,y,z)^T)$.

   (b) Let V be a finite-dimensional vector space over F with $\dim(V) = n$. Let $\mathcal{B} = \langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$ be an ordered basis of V. Compute $\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id})$, the matrix representation of the identity map $\mathrm{id} : V \to V$, and verify that it's equal to I_n. [See Lecture 13 Warmup #3.]

   (c) Use the fact that I_n is the encoding of the identity map in any ordered basis to explain why $XI_\ell = X$ and $I_kX = X$ for any $X \in M_{k,\ell}(F)$.

   (d) Verify 3c specifically for the following example, where B and D are from Problem 1.

   Compute **(i)** I_2B, **(ii)** BI_1, **(iii)** I_2D, **(iv)** DI_3.

   (e) Note that $I_3 = E_{1,1} + E_{2,2} + E_{3,3}$. Reconcile your answers to 2b with the fact that $I_3X = X$ and $XI_3 = X$ for all $X \in M_3(F)$.

   (f) **CAUTION!!** The identity function is only represented by the identity matrix when the domain and codomain bases are the same.[2] Consider the following ordered bases of $\mathbb{R}^3$:
$$\mathcal{E} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \quad \mathcal{A} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \quad \text{and} \quad \mathcal{B} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$
   Compute the following. **(i)** $\mathrm{Rep}_{\mathcal{E}}^{\mathcal{A}}(\mathrm{id})$, **(ii)** $\mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(\mathrm{id})$, **(iii)** $\mathrm{Rep}_{\mathcal{E}}^{\mathcal{B}}(\mathrm{id})$, **(iv)** $\mathrm{Rep}_{\mathcal{B}}^{\mathcal{E}}(\mathrm{id})$.

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[1] Your answers may depend on whether some of i, j, k, ℓ are equal or not. "Describe" might be something like "the $n \times n$ matrix whose ith column is. . . ."

[2] I promise that there will be good reasons to study the identity map expressed in mixed bases.