Lecture 13 exercises

1. Let
$$A = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & -1 \end{pmatrix}.$$

   (a) For each of the following, decide whether or not the product is defined. If so, compute it; if not, say why not. **(i)** AB, **(ii)** BA, **(iii)** AC, **(iv)** CA, **(v)** BC, **(vi)** CB, **(vii)** CD, **(viii)** DC.

   (b) Compare $(CA)B$ (multiply CA and B) and $C(AB)$ (multiply C and AB).

   (c) For $n \in \mathbb{Z}_{\geq 1}$, we denote $A^\ell := \overbrace{AA\cdots A}^{\ell \text{ terms}}$. For each of the following, decide whether or not the product is defined. If so, compute it; if not, say why not. **(i)** A^2 **(ii)** B^2 **(iii)** C^2 **(iv)** D^2

2. Recall $E_{i,j}$ denotes a matrix with a 1 in row i, col j and 0's elsewhere.

   (a) Working in $M_3(F) = M_{3,3}(F)$, let
$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{pmatrix}.$$
   Compute **(i)** $E_{1,2}X$, **(ii)** $XE_{1,2}$, **(iii)** $E_{3,3}X$, **(iv)** $XE_{3,3}$.

   (b) Working more generally over $M_n(F) = M_{n,n}(F)$, let $X \in M_n(F)$ and let $1 \leq i,j,k,\ell \leq n$. Describe/conjecture the following.[1] **(i)** $E_{i,j}X$, **(ii)** $XE_{i,j}$, **(iii)** $E_{i,j}XE_{k,\ell}$, **(iv)** $E_{i,j}E_{k,\ell}$.

3. The **identity matrix** I_n is the $n \times n$ matrix with 1 and (i,i)-entry for $i = 1, \ldots, n$, and 0's elsewhere. For example,
$$I_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) If $f : F^3 \to F^3$ is the function associated to I_3, compute $f((x,y,z)^T)$.

   (b) Let V be a finite-dimensional vector space over F with $\dim(V) = n$. Let $\mathcal{B} = \langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$ be an ordered basis of V. Compute $\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id})$, the matrix representation of the identity map $\mathrm{id} : V \to V$, and verify that it's equal to I_n. [See warmup #3.]

   (c) Use the fact that I_n is the encoding of the identity map in any ordered basis to explain why $XI_\ell = X$ and $I_kX = X$ for any $X \in M_{k,\ell}(F)$.

   (d) Verify 3c specifically for the following example, where B and D are from Problem 1.

   Compute **(i)** I_2B, **(ii)** BI_1, **(iii)** I_2D, **(iv)** DI_3.

   (e) Note that $I_3 = E_{1,1} + E_{2,2} + E_{3,3}$. Reconcile your answers to 2b with the fact that $I_3X = X$ and $XI_3 = X$ for all $X \in M_3(F)$.

   (f) **CAUTION!!** The identity function is only represented by the identity matrix when the domain and codomain bases are the same.[2] Consider the following ordered bases of $\mathbb{R}^3$:
$$\mathcal{E} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \quad \mathcal{A} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \quad \text{and} \quad \mathcal{B} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$
   Compute the following. **(i)** $\mathrm{Rep}_{\mathcal{E}}^{\mathcal{A}}(\mathrm{id})$, **(ii)** $\mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(\mathrm{id})$, **(iii)** $\mathrm{Rep}_{\mathcal{E}}^{\mathcal{B}}(\mathrm{id})$, **(iv)** $\mathrm{Rep}_{\mathcal{B}}^{\mathcal{E}}(\mathrm{id})$.

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[1] Your answers may depend on whether some of i,j,k,ℓ are equal or not. "Describe" might be something like "the $n \times n$ matrix whose ith column is...."

[2] I promise that there will be good reasons to study the identity map expressed in mixed bases.