Lecture 13: Matrix multiplication

Warmup. For each of the following, think of $\alpha_i, \beta_i, \gamma_i, \delta_i, \omega_i \in F$ as constants.

1. Let
$$f\colon \mathbb{R}^2 \mapsto \mathbb{R}^2 \quad \text{defined by} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \end{pmatrix}, \quad \text{and}$$
$$g\colon \mathbb{R}^2 \mapsto \mathbb{R}^2 \quad \text{defined by} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \gamma_1 x + \delta_1 y \\ \gamma_2 x + \delta_2 y \end{pmatrix}.$$
   (a) Compute $(g \circ f)((x,y)^T)$. [This is just function composition—nothing fancy. Plug $f((x,y)^T)$ into g and simplify/collect like terms.]
   (b) Write down the matrices representations of f, g, $g \circ f$, all with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$.
2. Let
$$f\colon \mathbb{R}^2 \mapsto \mathbb{R}^3 \quad \text{defined by} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \\ \alpha_3 x + \beta_3 y \end{pmatrix}, \quad \text{and}$$
$$g\colon \mathbb{R}^3 \mapsto \mathbb{R}^2 \quad \text{defined by} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \gamma_1 x + \delta_1 y + \omega_1 z \\ \gamma_2 x + \delta_2 y + \omega_2 z \end{pmatrix}.$$
   (a) Compute $(g \circ f)((x,y)^T)$ and $(f \circ g)((x,y,z)^T)$.
   (b) Write down the matrices representations of f, g, $g \circ f$, $f \circ g$, all with respect to the standard bases $\mathcal{E}_2 = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathcal{E}_3 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
3. For any vector space, we have $\mathrm{id} : V \to V$ sending $\mathrm{id} : \mathbf{v} \mapsto \mathbf{v}$. For each of the following vector spaces V and ordered bases $\mathcal{B}$, compute $\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id})$.
   (a) $V = \mathbb{R}_2$, $\mathcal{B} = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$, (b) $V = \mathbb{R}_2$, $\mathcal{B} = \langle (1,1)^T, (3,5)^T \rangle$,
   (c) $V = \mathcal{P}_3(F)$, $\mathcal{B} = \langle 1, x, x^2, x^3, x^4 \rangle$,
   (d) $V = \mathcal{P}_3(F)$, $\mathcal{B} = \langle 1 + 2x, 1 - x + x^3, 3 - 2x, 1 + x + x^4 \rangle$.

(See last pages for solutions to 1 & 2)

Recall: The set of linear functions $f : F^\ell \to F^k$ are in bijection with $k \times \ell$ matrices: for

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,\ell} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,\ell} \end{pmatrix} \in M_{k,\ell}(F),$$

we associate a function $A : F^\ell \to F^k$ given by

$$A\colon \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_\ell \end{pmatrix} \mapsto \begin{pmatrix} (a_{1,1}, \dots, a_{1,\ell}) \cdot (v_1, \dots, v_\ell) \\ (a_{2,1}, \dots, a_{2,\ell}) \cdot (v_1, \dots, v_\ell) \\ \vdots \\ (a_{k,1}, \dots, a_{k,\ell}) \cdot (v_1, \dots, v_\ell) \end{pmatrix} \quad \text{i.e.} \quad A\colon \mathbf{v} \mapsto \begin{pmatrix} \mathbf{row}_1 \cdot \mathbf{v}^T \\ \mathbf{row}_2 \cdot \mathbf{v}^T \\ \vdots \\ \mathbf{row}_k \cdot \mathbf{v}^T \end{pmatrix}.$$

We saw that

$$A : \mathbf{e}_i \mapsto \mathbf{col}_i,$$

the ith column of A.

Question. Given $A : F^m \to F^\ell$ and $B : F^\ell \to F^k$, how do we compute the matrix associated to $B \circ A$? **Notation:** We write $BA := B \circ A$.

Answer:
Compute the image of $\mathbf{e}_j$, and insert the result into the jth column of BA.

Question. Given $A : F^m \to F^\ell$ and $B : F^\ell \to F^k$, how do we compute the matrix associated to $B \circ A$? **Notation:** We write $BA := B \circ A$.

Answer:
Compute the image of $\mathbf{e}_j$, and insert the result into the jth column of BA.

Note that

$$A : F^m \to F^\ell \text{ means } A \in M_{\ell,m}(F)$$

and

$$B : F^\ell \to F^k \text{ means } B \in M_{k,\ell}(F).$$

[Reality check: $BA : F^m \xrightarrow{A} F^\ell \xrightarrow{B} F^k$, so $BA \in M_{k,m}$.]

Now, to compute the function BA, let's compute the image of $\mathbf{e}_j$ for $j = 1, \dots, m$ (the result is the jth column of BA):

$$A\mathbf{e}_j = \mathbf{col}_j(A); \quad \text{so that} \quad BA\mathbf{e}_j = B(A\mathbf{e}_j) = B(\mathbf{col}_j(A))$$

$$= \begin{pmatrix} \mathbf{row}_1(B) \cdot (\mathbf{col}_j(A))^T \\ \mathbf{row}_2(B) \cdot (\mathbf{col}_j(A))^T \\ \vdots \\ \mathbf{row}_k(B) \cdot (\mathbf{col}_j(A))^T \end{pmatrix}$$

In particular, the entry in the ith row and jth column of BA (with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m$) is

$$\boxed{(BA)_{i,j} = \mathbf{row}_i(B) \cdot (\mathbf{col}_j(A))^T.}$$

Example:

$$\underbrace{\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 1 & -1 \end{pmatrix}}_{3 \times 4} \underbrace{\begin{pmatrix} -4 & 0 \\ 1 & 1 \\ 0 & 0 \\ 3 & 2 \end{pmatrix}}_{4 \times 2} = \underbrace{\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{pmatrix}}_{3 \times 2}$$

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 1 & 1 \\ 0 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & \alpha_{1,2} \\ \mathbf{3} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{pmatrix}$$

$$\alpha_{1,1} = (1, -1, 2, 3) \cdot (-4, 1, 0, 3) = 1(-4) + -1(1) + 2(0) + 3(3) = 4;$$

$$\alpha_{2,1} = (0, 3, 1, 0) \cdot (-4, 1, 0, 3) = 0(-4) + 3(1) + 1(0) + 0(3) = \mathbf{3};$$

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 1 & 1 \\ 0 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & \mathbf{5} \\ 3 & \alpha_{2,2} \\ -23 & \alpha_{3,2} \end{pmatrix}$$

$$\begin{aligned}
\alpha_{1,1} &= (1,-1,2,3)\cdot(-4,1,0,3) &&= 1(-4)+-1(1)+2(0)+3(3) &&= 4;\\
\alpha_{2,1} &= (0,3,1,0)\cdot(-4,1,0,3) &&= 0(-4)+3(1)+1(0)+0(3) &&= 3;\\
\alpha_{3,1} &= (5,0,1,-1)\cdot(-4,1,0,3) &&= 5(-4)+0(1)+1(0)+-1(3) &&= -23;\\
\alpha_{1,2} &= (1,-1,2,3)\cdot(0,1,0,2) &&= 1(0)+-1(1)+2(0)+3(2) &&= \mathbf{5};
\end{aligned}$$

Example:

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 1 & 1 \\ 0 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 3 & 3 \\ -23 & \mathbf{-2} \end{pmatrix}$$

$$\begin{aligned}
\alpha_{1,1} &= (1,-1,2,3)\cdot(-4,1,0,3) &&= 1(-4)+-1(1)+2(0)+3(3) &&= 4;\\
\alpha_{2,1} &= (0,3,1,0)\cdot(-4,1,0,3) &&= 0(-4)+3(1)+1(0)+0(3) &&= 3;\\
\alpha_{3,1} &= (5,0,1,-1)\cdot(-4,1,0,3) &&= 5(-4)+0(1)+1(0)+-1(3) &&= -23;\\
\alpha_{1,2} &= (1,-1,2,3)\cdot(0,1,0,2) &&= 1(0)+-1(1)+2(0)+3(2) &&= 5;\\
\alpha_{2,2} &= (0,3,1,0)\cdot(0,1,0,2) &&= 0(0)+3(1)+1(0)+0(2) &&= 3;\\
\alpha_{3,2} &= (5,0,1,-1)\cdot(0,1,0,2) &&= 5(0)+0(1)+1(0)+-1(2) &&= \mathbf{-2}.
\end{aligned}$$

Matrix multiplication

Just like we define addition and scaling of matrices to agree with addition and scaling of the associated functions, we define a "product" on matrices that agrees with composition of the associated functions.

Definition. For matrices $X \in M_{k,\ell}(F)$ and $Y \in M_{m,n}(F)$, if $\ell = m$ we define the **product** of X and Y to be $XY \in M_{k,n}$, the matrix with (i,j)-entry (meaning the entry in row i and column j) to be

$$(XY)_{i,j} = \mathbf{row}_i(X) \cdot (\mathbf{col}_j(Y))^T = \sum_{r=1}^{\ell} X_{i,r} Y_{r,j}.$$

If $\ell \neq m$, we say XY is **undefined**. (Just like function composition is only defined when the domain/codomain match up appropriately.)

Since matrix multiplication is really just *function composition*, we have already shown that it satisfies the following:

1. Left distributive: $X(Y+Z) = XY + XZ$
   for any $X \in M_{k,\ell}(F)$ and $Y, Z \in M_{\ell,n}(F)$;
2. Right distributive: $(X+Y)Z = XZ + YZ$
   for any $X, Y \in M_{k,\ell}(F)$ and $Z \in M_{\ell,n}(F)$;
3. Associative: $X(YZ) = (XY)Z$
   for any $X \in M_{k,\ell}(F)$, $Y \in M_{\ell,m}(F)$, and $Z \in M_{m,n}(F)$.

Lecture 13 exercises

1. Let
$$A = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & -1 \end{pmatrix}.$$

   (a) For each of the following, decide whether or not the product is defined. If so, compute it; if not, say why not. **(i)** AB, **(ii)** BA, **(iii)** AC, **(iv)** CA, **(v)** BC, **(vi)** CB, **(vii)** CD, **(viii)** DC.

   (b) Compare $(CA)B$ (multiply CA and B) and $C(AB)$ (multiply C and AB).

   (c) For $n \in \mathbb{Z}_{\geq 1}$, we denote $A^\ell := \overbrace{AA\cdots A}^{\ell \text{ terms}}$. For each of the following, decide whether or not the product is defined. If so, compute it; if not, say why not. **(i)** A^2 **(ii)** B^2 **(iii)** C^2 **(iv)** D^2

2. Recall $E_{i,j}$ denotes a matrix with a 1 in row i, col j and 0's elsewhere.

   (a) Working in $M_3(F) = M_{3,3}(F)$, let
$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{pmatrix}.$$
   Compute **(i)** $E_{1,2}X$, **(ii)** $XE_{1,2}$, **(iii)** $E_{3,3}X$, **(iv)** $XE_{3,3}$.

   (b) Working more generally over $M_n(F) = M_{n,n}(F)$, let $X \in M_n(F)$ and let $1 \leq i, j, k, \ell \leq n$. Describe/conjecture the following.[1] **(i)** $E_{i,j}X$, **(ii)** $XE_{i,j}$, **(iii)** $E_{i,j}XE_{k,\ell}$, **(iv)** $E_{i,j}E_{k,\ell}$.

3. The **identity matrix** I_n is the $n \times n$ matrix with 1 and (i,i)-entry for $i = 1, \ldots, n$, and 0's elsewhere. For example,
$$I_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) If $f : F^3 \to F^3$ is the function associated to I_3, compute $f((x,y,z)^T)$.

   (b) Let V be a finite-dimensional vector space over F with $\dim(V) = n$. Let $\mathcal{B} = \langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$ be an ordered basis of V. Compute $\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id})$, the matrix representation of the identity map $\mathrm{id} : V \to V$, and verify that it's equal to I_n. [See warmup #3.]

   (c) Use the fact that I_n is the encoding of the identity map in any ordered basis to explain why $XI_\ell = X$ and $I_kX = X$ for any $X \in M_{k,\ell}(F)$.

   (d) Verify 3c specifically for the following example, where B and D are from Problem 1.

   Compute **(i)** I_2B, **(ii)** BI_1, **(iii)** I_2D, **(iv)** DI_3.

   (e) Note that $I_3 = E_{1,1} + E_{2,2} + E_{3,3}$. Reconcile your answers to 2b with the fact that $I_3X = X$ and $XI_3 = X$ for all $X \in M_3(F)$.

   (f) **CAUTION!!** The identity function is only represented by the identity matrix when the domain and codomain bases are the same.[2] Consider the following ordered bases of $\mathbb{R}^3$:
$$\mathcal{E} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \quad \mathcal{A} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \quad \text{and} \quad \mathcal{B} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$
   Compute the following. **(i)** $\mathrm{Rep}_{\mathcal{E}}^{\mathcal{A}}(\mathrm{id})$, **(ii)** $\mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(\mathrm{id})$, **(iii)** $\mathrm{Rep}_{\mathcal{E}}^{\mathcal{B}}(\mathrm{id})$, **(iv)** $\mathrm{Rep}_{\mathcal{B}}^{\mathcal{E}}(\mathrm{id})$.

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[1] Your answers may depend on whether some of i, j, k, ℓ are equal or not. "Describe" might be something like "the $n \times n$ matrix whose ith column is. . . "

[2] I promise that there will be good reasons to study the identity map expressed in mixed bases.

Solutions to warmup 1 & 2

1. $$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \end{pmatrix}$$ is associated to the matrix $A = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$; and

$$g : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \gamma_1 x + \delta_1 y \\ \gamma_2 x + \delta_2 y \end{pmatrix}$$ is associated to the matrix $B = \begin{pmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix}$.

So

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{f} \begin{pmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \end{pmatrix} \xmapsto{g} \begin{pmatrix} \gamma_1(\alpha_1 x + \beta_1 y) + \delta_1(\alpha_2 x + \beta_2 y) \\ \gamma_2(\alpha_1 x + \beta_1 y) + \delta_2(\alpha_2 x + \beta_2 y) \end{pmatrix}$$

$$= \begin{pmatrix} (\gamma_1\alpha_1 + \delta_1\alpha_2)x + (\gamma_1\beta_1 + \delta_1\beta_2)y \\ (\gamma_2\alpha_1 + \delta_2\alpha_2)x + (\gamma_2\beta_1 + \delta_2\beta_2)y \end{pmatrix}$$

$$= \begin{pmatrix} \big((\gamma_1, \delta_1) \cdot (\alpha_1, \alpha_2)\big)x + \big((\gamma_1, \delta_1) \cdot (\beta_1, \beta_2)\big)y \\ \big((\gamma_2, \delta_2) \cdot (\alpha_1, \alpha_2)\big)x + \big((\gamma_2, \delta_2) \cdot (\beta_1, \beta_2)\big)y \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{row}_1(B) \cdot \mathbf{col}_1(A)^T)x + (\mathbf{row}_1(B) \cdot \mathbf{col}_2(A)^T)y \\ (\mathbf{row}_2(B) \cdot \mathbf{col}_1(A)^T)x + (\mathbf{row}_2(B) \cdot \mathbf{col}_2(A)^T)y \end{pmatrix}.$$

So $g \circ f$ is associated to the matrix

$$B \circ A = \begin{pmatrix} \mathbf{row}_1(B) \cdot \mathbf{col}_1(A)^T & \mathbf{row}_1(B) \cdot \mathbf{col}_2(A)^T \\ \mathbf{row}_2(B) \cdot \mathbf{col}_1(A)^T & \mathbf{row}_2(B) \cdot \mathbf{col}_2(A)^T \end{pmatrix}$$

2.

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \\ \alpha_3 x + \beta_3 y \end{pmatrix}$$ is assoc. to the matrix $A = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}$; and

$$g : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \gamma_1 x + \delta_1 y + \omega_1 z \\ \gamma_2 x + \delta_2 y + \omega_2 z \end{pmatrix}$$ is assoc. to the matrix $B = \begin{pmatrix} \gamma_1 & \delta_1 & \omega_1 \\ \gamma_2 & \delta_2 & \omega_2 \end{pmatrix}$.

So

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{f} \begin{pmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \\ \alpha_3 x + \beta_3 y \end{pmatrix} \xmapsto{g} \begin{pmatrix} \gamma_1(\alpha_1 x + \beta_1 y) + \delta_1(\alpha_2 x + \beta_2 y) + \omega_1(\alpha_3 x + \beta_3 y) \\ \gamma_2(\alpha_1 x + \beta_1 y) + \delta_2(\alpha_2 x + \beta_2 y) + \omega_2(\alpha_3 x + \beta_3 y) \end{pmatrix}$$

$$= \begin{pmatrix} (\gamma_1\alpha_1 + \delta_1\alpha_2 + \omega_1\alpha_3)x + (\gamma_1\beta_1 + \delta_1\beta_2 + \omega_1\beta_3)y \\ (\gamma_2\alpha_1 + \delta_2\alpha_2 + \omega_2\alpha_3)x + (\gamma_2\beta_1 + \delta_2\beta_2 + \omega_2\beta_3)y \end{pmatrix}$$

$$= \begin{pmatrix} \big((\gamma_1, \delta_1, \omega_1) \cdot (\alpha_1, \alpha_2, \alpha_3)\big)x + \big((\gamma_1, \delta_1, \omega_1) \cdot (\beta_1, \beta_2, \beta_3)\big)y \\ \big((\gamma_2, \delta_2, \omega_2) \cdot (\alpha_1, \alpha_2, \alpha_3)\big)x + \big((\gamma_2, \delta_2, \omega_2) \cdot (\beta_1, \beta_2, \beta_3)\big)y \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{row}_1(B) \cdot \mathbf{col}_1(A)^T)x + (\mathbf{row}_1(B) \cdot \mathbf{col}_2(A)^T)y \\ (\mathbf{row}_2(B) \cdot \mathbf{col}_1(A)^T)x + (\mathbf{row}_2(B) \cdot \mathbf{col}_2(A)^T)y \end{pmatrix}.$$

So $g \circ f$ is associated to the matrix

$$B \circ A = \begin{pmatrix} \mathbf{row}_1(B) \cdot \mathbf{col}_1(A)^T & \mathbf{row}_1(B) \cdot \mathbf{col}_2(A)^T \\ \mathbf{row}_2(B) \cdot \mathbf{col}_1(A)^T & \mathbf{row}_2(B) \cdot \mathbf{col}_2(A)^T \end{pmatrix}.$$

And

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{g} \begin{pmatrix} \gamma_1 x + \delta_1 y + \omega_1 z \\ \gamma_2 x + \delta_2 y + \omega_2 z \end{pmatrix}$$

$$\xmapsto{f} \begin{pmatrix} \alpha_1(\gamma_1 x + \delta_1 y + \omega_1 z) + \beta_1(\gamma_2 x + \delta_2 y + \omega_2 z) \\ \alpha_2(\gamma_1 x + \delta_1 y + \omega_1 z) + \beta_2(\gamma_2 x + \delta_2 y + \omega_2 z) \\ \alpha_3(\gamma_1 x + \delta_1 y + \omega_1 z) + \beta_3(\gamma_2 x + \delta_2 y + \omega_2 z) \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha_1\gamma_1 + \beta_1\gamma_2)x + (\alpha_1\delta_1 + \beta_1\delta_2)y + (\alpha_1\omega_1 + \beta_1\omega_2)z \\ (\alpha_2\gamma_1 + \beta_2\gamma_2)x + (\alpha_2\delta_1 + \beta_2\delta_2)y + (\alpha_2\omega_1 + \beta_2\omega_2)z \\ (\alpha_3\gamma_1 + \beta_3\gamma_2)x + (\alpha_3\delta_1 + \beta_3\delta_2)y + (\alpha_3\omega_1 + \beta_3\omega_2)z \end{pmatrix}$$

$$= \begin{pmatrix} \big((\alpha_1, \beta_1)\cdot(\gamma_1, \gamma_2)\big)x + \big((\alpha_1, \beta_1)\cdot(\delta_1, \delta_2)\big)y + \big((\alpha_1, \beta_1)\cdot(\omega_1, \omega_2)\big)z \\ \big((\alpha_2, \beta_2)\cdot(\gamma_1, \gamma_2)\big)x + \big((\alpha_2, \beta_2)\cdot(\delta_1, \delta_2)\big)y + \big((\alpha_2, \beta_2)\cdot(\omega_1, \omega_2)\big)z \\ \big((\alpha_3, \beta_3)\cdot(\gamma_1, \gamma_2)\big)x + \big((\alpha_3, \beta_3)\cdot(\delta_1, \delta_2)\big)y + \big((\alpha_3, \beta_3)\cdot(\omega_1, \omega_2)\big)z \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{row}_1(A)\cdot\mathbf{col}_1(B)^T)x + (\mathbf{row}_1(A)\cdot\mathbf{col}_2(B)^T)y + (\mathbf{row}_1(A)\cdot\mathbf{col}_3(B)^T)z \\ (\mathbf{row}_2(A)\cdot\mathbf{col}_1(B)^T)x + (\mathbf{row}_2(A)\cdot\mathbf{col}_2(B)^T)y + (\mathbf{row}_2(A)\cdot\mathbf{col}_3(B)^T)z \\ (\mathbf{row}_3(A)\cdot\mathbf{col}_1(B)^T)x + (\mathbf{row}_3(A)\cdot\mathbf{col}_2(B)^T)y + (\mathbf{row}_3(A)\cdot\mathbf{col}_3(B)^T)z \end{pmatrix}$$

So $f \circ g$ is associated to the matrix

$$A \circ B = \begin{pmatrix} (\mathbf{row}_1(A)\cdot\mathbf{col}_1(B)^T) & (\mathbf{row}_1(A)\cdot\mathbf{col}_2(B)^T) & (\mathbf{row}_1(A)\cdot\mathbf{col}_3(B)^T) \\ (\mathbf{row}_2(A)\cdot\mathbf{col}_1(B)^T) & (\mathbf{row}_2(A)\cdot\mathbf{col}_2(B)^T) & (\mathbf{row}_2(A)\cdot\mathbf{col}_3(B)^T) \\ (\mathbf{row}_3(A)\cdot\mathbf{col}_1(B)^T) & (\mathbf{row}_3(A)\cdot\mathbf{col}_2(B)^T) & (\mathbf{row}_3(A)\cdot\mathbf{col}_3(B)^T) \end{pmatrix}.$$