Lecture 12: Matrices as linear functions

Warmup. Let $A = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 2 & -1 \end{pmatrix}$.

Recall that A defines a function $A : F^3 \mapsto F^2$, given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (0,1,3)\cdot(x,y,z) \\ (-1,2,-1)\cdot(x,y,z) \end{pmatrix} = \begin{pmatrix} 0x + 1y + 3z \\ -1x + 2y - 1z \end{pmatrix}.$$

(a) Compute $A\mathbf{v}$ for $\mathbf{v} = (1,0,1)^T$, $\mathbf{v} = (-1,2,-3)^T$, and $\mathbf{v} = (0,0,0)^T$.

(b) Recall that the standard (ordered) basis of F^3 is

$$\mathcal{E} = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle, \quad \text{where} \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Compute $A\mathbf{e}$ for each $\mathbf{e} \in \mathcal{E}$.

(c) Compute $\mathcal{N}(A)$ (the vectors $\mathbf{v} \in F^3$ for which $A\mathbf{v} = \mathbf{0}_{F^2}$).

(d) What did you notice about your answer to part (b)? Can you prove your answer for a general matrix?

(e) How does the range of the function A relate to spaces we've studied before having to do with matrices?

(f) How does the rank of the function A relate to statistics we've studied before having to do with matrices?

(g) How does the nullspace of A relate spaces we've studied before having to do with matrices? nullity?

Recall: The set of functions $\mathcal{F}\mathrm{n}(\mathbb{R},\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ is a vector space over $\mathbb{R}$, where addition and scaling are defined **point-wise**: for all $x \in \mathbb{R}$, we define

$$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda(f(x)), \qquad (\star)$$

for $f, g \in \mathcal{F}\mathrm{n}(\mathbb{R},\mathbb{R})$ and $\lambda \in \mathbb{R}$. In fact, for any set X and any vector space V over a field F, the set of functions

$$\mathcal{F}\mathrm{n}(X,V) = \{f : X \to V\}$$

is a vector space over F using point-wise addition and scaling (i.e. using $(\star)$): I know how to add and scale elements of $f(X)$ *because they're elements of V*.

Now, for any vector spaces U and V over a field F, the set of linear functions (a.k.a. homomorphisms)

$$\mathrm{Hom}(U,V) = \{h : U \to V \mid h \text{ is linear }\}$$

is a subspace of $\mathcal{F}\mathrm{n}(U,V)$ because

- the zero map, $\zeta : \mathbf{u} \mapsto 0$ for all $\mathbf{u} \in U$, is linear; and
- if $f, g : U \to V$ is linear and $\lambda \in F$, then

$$f + g \quad \text{and} \quad \lambda f \quad \text{are both linear.}^{*}$$

* You must check:

$$(f+g)(\mathbf{u}_1 + \mathbf{u}_2) = (f+g)(\mathbf{u}_1) + (f+g)(\mathbf{u}_2), \qquad (f+g)(\alpha\mathbf{u}) = \alpha(f+g)(\mathbf{u}),$$
$$(\lambda f)(\mathbf{u}_1 + \mathbf{u}_2) = (\lambda f)(\mathbf{u}_1) + (\lambda f))(\mathbf{u}_2), \quad \text{and} \quad (\lambda f))(\alpha\mathbf{u}) = \alpha(\lambda f))(\mathbf{u}),$$

for all $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in U$ and $\alpha \in V$. Plug in and follow your nose.

Last time

To a matrix $A \in M_{k,\ell}(F)$,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,\ell} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,\ell} \end{pmatrix},$$

we associate a function $A : F^{\ell} \to F^{k}$ given by

$$A\colon \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_\ell \end{pmatrix} \mapsto \begin{pmatrix} (a_{1,1}, \dots, a_{1,\ell}) \cdot (v_1, \dots, v_\ell) \\ (a_{2,1}, \dots, a_{2,\ell}) \cdot (v_1, \dots, v_\ell) \\ \vdots \\ (a_{k,1}, \dots, a_{k,\ell}) \cdot (v_1, \dots, v_\ell) \end{pmatrix} \quad \text{i.e.} \quad A\colon \mathbf{v} \mapsto \begin{pmatrix} \mathbf{row}_1 \cdot \mathbf{v}^T \\ \mathbf{row}_2 \cdot \mathbf{v}^T \\ \vdots \\ \mathbf{row}_k \cdot \mathbf{v}^T \end{pmatrix},$$

where $\mathbf{row}_i$ is the ith row vector of A, and $\mathbf{v}^T$ is the **transpose** of $\mathbf{v}$ (from Homework 4), and $\mathbf{row}_i \cdot \mathbf{v}^T$ is the **dot product**, given by

$$(c_1, c_2, \dots, c_n) \cdot (d_1, d_2, \dots, d_n) := c_1 d_1 + c_2 d_2 + \cdots + c_n d_n.$$

Notation: write $A\mathbf{v}$ to mean $A(\mathbf{v})$. $\longleftarrow$ $\boxed{\text{Why is this good notation??}}$

Any time we write something using notation that is evocative of multiplication, I do so because it has algebraic structure that has similar properties as multiplication.

1. Lemma: $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, (for all $\mathbf{x}, \mathbf{y} \in F^\ell$).
   Pf. The ith row in $A(\mathbf{x}+\mathbf{y})$ is
   $$\mathbf{r}_i \cdot (\mathbf{x}+\mathbf{y})^T = \mathbf{r}_i \cdot (\mathbf{x}^T + \mathbf{y}^T) = \mathbf{r}_i \cdot \mathbf{x}^T + \mathbf{r}_i \cdot \mathbf{y}^T.$$

2. Lemma: $(A+B)(\mathbf{x}) = A\mathbf{x} + B\mathbf{x}$, (for all $A, B \in M_{k,\ell}(F)$ and $\mathbf{x} \in F^\ell$).
   Pf. Let $\mathbf{s}_1, \ldots, \mathbf{s}_k$ be the row vectors of B. Then the ith row vector of $A+B$ is $\mathbf{r}_i + \mathbf{s}_i$. So the ith row vector of $(A+B)\mathbf{x}$ is
   $$(\mathbf{r}_i + \mathbf{s}_i) \cdot \mathbf{x}^T = \mathbf{r}_i \cdot \mathbf{x}^T + \mathbf{s}_i \cdot \mathbf{x}^T.$$

3. Lemma: $A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = (\lambda A)\mathbf{x}$, (for all $\mathbf{x} \in F^\ell$ and $\lambda \in F$).
   Pf. The ith row in $A(\lambda\mathbf{x})$ is
   $$\mathbf{r}_i \cdot (\lambda\mathbf{x})^T = \mathbf{r}_i \cdot (\lambda\mathbf{x}^T) = \lambda(\mathbf{r}_i \cdot \mathbf{x}^T) = (\lambda\mathbf{r}_i) \cdot \mathbf{x}^T.$$

Lemma. For all $A, B \in M_{k,\ell}$, $\mathbf{x}, \mathbf{y} \in F^\ell$, and $\lambda \in F$, we have

1. $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$,
2. $(A \underset{\substack{\uparrow \\ \text{coordinate-wise} \\ \text{addition of matrices}}}{+} B)(\mathbf{x}) = A\mathbf{x} \underset{\substack{\nwarrow \\ \text{coordinate-wise} \\ \text{addition of vectors}}}{+} B\mathbf{x}$, and
3. $A(\underset{\substack{\nearrow \\ \text{vector} \\ \text{scaling} \\ \text{in } F^\ell}}{\lambda}\mathbf{x}) \overset{(a)}{=} \underset{\substack{\uparrow \\ \text{vector} \\ \text{scaling} \\ \text{in } F^k}}{\lambda}(A\mathbf{x}) \overset{(b)}{=} (\underset{\substack{\nwarrow \\ \text{matrix} \\ \text{scaling} \\ \text{in } M_{k,\ell}(F)}}{\lambda}A)\mathbf{x}$

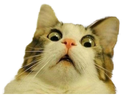
Note that 1 and 3(a) prove that A, as a function, is linear! 

Further, 2 and 3(b) say that the vector space structure on $M_{k,\ell}(F)$ is *also* preserved when we start thinking about $M_{k,\ell}(F)$ as functions!

Namely the map

$$\Phi : M_{k,\ell}(F) \longrightarrow \mathrm{Hom}(F^\ell, F^k) = \{\text{ linear functions } h : F^\ell \to F^k\}$$

$$A \longmapsto \text{“interpret } A \text{ as the function } A\colon \mathbf{v} \mapsto \begin{pmatrix} \mathbf{row}_1 \cdot \mathbf{v}^T \\ \mathbf{row}_2 \cdot \mathbf{v}^T \\ \vdots \\ \mathbf{row}_k \cdot \mathbf{v}^T \end{pmatrix}\text{”}$$

is, itself, a linear function!!  [**Question:** Is it an isomorphism?]

Back to linear extensions

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,\ell} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{1,\ell} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{1,\ell} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,\ell} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_\ell \end{pmatrix} = \begin{pmatrix} (a_{1,1}, a_{1,2}, a_{1,3}, \dots, a_{1,\ell}) \cdot (v_1, v_2, v_3, \dots, v_\ell) \\ (a_{2,1}, a_{2,2}, a_{2,3}, \dots, a_{2,\ell}) \cdot (v_1, v_2, v_3, \dots, v_\ell) \\ (a_{2,1}, a_{3,2}, a_{3,3}, \dots, a_{2,\ell}) \cdot (v_1, v_2, v_3, \dots, v_\ell) \\ \vdots \\ (a_{k,1}, a_{k,2}, a_{k,3}, \dots, a_{k,\ell}) \cdot (v_1, v_2, v_3, \dots, v_\ell) \end{pmatrix}$$

For $A \in M_{k,\ell}(F)$, what does A do to $\mathcal{E} = \langle \mathbf{e}_1, \dots, \mathbf{e}_\ell \rangle$?

Note that

$$\begin{aligned}
(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\ell) \cdot (\mathbf{1}, 0, 0, \dots, 0) &= \alpha_1 \mathbf{1} + \alpha_2 0 + \alpha_3 0 + \cdots + \alpha_\ell 0 = \alpha_1, \\
(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\ell) \cdot (0, \mathbf{1}, 0, \dots, 0) &= \alpha_1 0 + \alpha_2 \mathbf{1} + \alpha_3 0 + \cdots + \alpha_\ell 0 = \alpha_2, \\
(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\ell) \cdot (0, 0, \mathbf{1}, \dots, 0) &= \alpha_1 0 + \alpha_2 0 + \alpha_3 \mathbf{1} + \cdots + \alpha_\ell 0 = \alpha_3, \\
&\vdots \\
(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\ell) \cdot (0, 0, 0, \dots, \mathbf{1}) &= \alpha_1 0 + \alpha_2 0 + \alpha_3 0 + \cdots + \alpha_\ell \mathbf{1} = \alpha_\ell;
\end{aligned}$$

i.e. $\boxed{(\alpha_1, \dots, \alpha_\ell) \cdot \mathbf{e}_i^T = \alpha_i.}$ So $\boxed{A\mathbf{e}_i = \mathbf{col}_i}$, the ith column vector of A.
Hence, A is the linear extension of the map

$$A : \mathcal{E} \to F^k \quad \text{given by} \quad \mathbf{e}_i \mapsto \mathbf{col}_i, \quad \text{for } i = 1, \dots, \ell.$$

Range = Column Space

For $A \in M_{k,\ell}(F)$, the associated function is

$$A : \mathbf{e}_i \mapsto \mathbf{col}_i \quad \text{for } i = 1, \dots, \ell, \quad \text{and extend linearly,} \qquad (\star)$$

where $\mathbf{col}_i$ is the ith column vector of A.

So the **range** of A is

$$\begin{aligned}
\mathcal{R}(A) &= \{A\mathbf{v} \mid \mathbf{v} \in F^\ell\} \\
&= \{A(x_1\mathbf{e}_1 + \cdots + x_\ell \mathbf{e}_\ell) \mid x_1, \dots, x_\ell \in F\}, && \text{since } \mathcal{E} \text{ is a basis of } F^\ell, \\
&= \{x_1(A\mathbf{e}_1) + \cdots + x_\ell(A\mathbf{e}_\ell) \mid x_1, \dots, x_\ell \in F\}, && \text{since } A \text{ is linear,} \\
&= \{x_1\mathbf{col}_1 + \cdots + x_\ell \mathbf{col}_\ell \mid x_1, \dots, x_\ell \in F\}, && \text{by } (\star), \\
&= \mathrm{ColSpace}(A).
\end{aligned}$$

Reality check: What space do we expect to find $\mathrm{ColSpace}(A)$ in? What is the codomain of A? Are they the same? [Yes: both are F^k]

Therefore,

$$\begin{aligned}
\mathrm{rank}(\text{the function } A) &= \dim(\mathcal{R}(\text{the function } A)) \\
&= \dim(\mathrm{ColSpace}(\text{the matrix } A)) = \mathrm{rank}(\text{the matrix } A).
\end{aligned}$$

(phew!!!)

Nullspace = Solution Space

For $A \in M_{k,\ell}(F)$, the associated function is

$$A : \mathbf{e}_i \mapsto \mathbf{col}_i \quad \text{for } i = 1, \ldots, \ell, \quad \text{and extend linearly,} \qquad (\star)$$

where $\mathbf{col}_i$ is the ith column vector of A.

So the **nullspace** of A is

$$\begin{aligned}
\mathcal{N}(A) &= \{\mathbf{x} \in F^\ell \mid A\mathbf{x} = \mathbf{0}\} \\
&= \{(x_1, \ldots, x_\ell)^T \in F^\ell \mid A(x_1\mathbf{e}_1 + \cdots + x_\ell\mathbf{e}_\ell) = \mathbf{0}\}, && \text{exp. in the basis } \mathcal{E}, \\
&= \{(x_1, \ldots, x_\ell)^T \in F^\ell \mid x_1(A\mathbf{e}_1) + \cdots + x_\ell(A\mathbf{e}_\ell) = \mathbf{0}\}, && \text{since } A \text{ is linear,} \\
&= \{(x_1, \ldots, x_\ell)^T \in F^\ell \mid x_1\mathbf{col}_1 + \cdots + x_\ell\mathbf{col}_\ell = \mathbf{0}\}, && \text{by } (\star), \\
&= \{ \text{ soln's to the homog. system associated to } A \ \}.
\end{aligned}$$

Reality check: What space do we expect to find the solution space in? What is the domain of A? Are they the same? [Yes: both are F^ℓ]

Therefore,

$$\begin{aligned}
\text{nullity}(\text{the function } A) &= \dim(\mathcal{N}(\text{the function } A)) \\
&= \dim(\text{SolutionSpace}(\text{the matrix } A)) && \textit{(phew!!!)} \\
&= \#(\text{cols of } A) - \text{rank}(\text{the matrix } A).
\end{aligned}$$

What about the row space???

Short answer:

If $F = \mathbb{R}$ or $\mathbb{Q}$, then the row space is exactly the piece of F^ℓ that maps bijectively onto $\mathcal{R}(A)$ in the proof of the Rank-Nullity Theorem.
(Hence $\dim(\text{RowSpace}(A)) = \text{rank}(A)$.)

If $F = \mathbb{C}$ or $\mathbb{0}$ or $\mathbb{F}_2$, then something more subtle is going on!
(Compare to HW4, #2.)

Long answer:

Someday, we'll talk about **inner products**, of which "dot product" is only one example. These will allow us to better analyze this question.

Take a moment and compare results
back to the example in the warmup!

Let U and V be vector spaces over a field F.

Remember: Given a fixed basis $\mathcal{B}$ of U, using linear extension,

defining a function $H : \mathcal{B} \to V$	*is exactly the same as*	defining a *linear* function $h : U \to V$.

Example: The function (see warmup)

$$\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle \to \mathbb{R}^2 \quad \text{defined by} \quad \begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\-1\end{pmatrix}, \ \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\2\end{pmatrix}, \ \text{and} \ \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}3\\-1\end{pmatrix}$$

linearly extends to the function

$$\mathbb{R}^3 \to \mathbb{R}^2 \quad \text{defined by} \quad \begin{pmatrix}x\\y\\z\end{pmatrix} \mapsto \begin{pmatrix}y+3z\\-x+2y-z\end{pmatrix};$$

which, in turn, restricts to the function

$$\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle \to \mathbb{R}^2 \quad \text{defined by} \quad \begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\-1\end{pmatrix}, \ \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\2\end{pmatrix}, \ \text{and} \ \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}3\\-1\end{pmatrix}.$$

Mathematically speaking

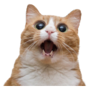
$$\mathcal{F}\mathrm{n}(\mathcal{B}, V) \xrightarrow{\text{linearly extend}} \mathrm{Hom}(U, V), \quad \text{and}$$

$$\mathcal{F}\mathrm{n}(\mathcal{B}, V) \xleftarrow{\text{restrict to } \mathcal{B} \subseteq U} \mathrm{Hom}(U, V)$$

are inverses of each other, and hence are bijections. Now, the linear function A is exactly the linear extension of the map $\mathbf{e}_i \mapsto \mathbf{col}_i$, so matrices are in bijection with linear functions!! (“Pick $f(\mathbf{e}_i)$’s” = “pick columns”.) Meaning,

$$\begin{aligned} \Phi : M_{k,\ell}(F) &\longrightarrow \mathrm{Hom}(F^\ell, F^k) = \{ \text{ linear functions } h : F^\ell \to F^k \} \\ A &\longmapsto \text{“interpret } A \text{ as a linear function”} \end{aligned}$$

is an isomorphism after all!!!!!!!!!!!!



[Remember: All finite-dimensional vector spaces are secretly F^n (for some n)!]

$$\mathrm{Rep}_{\mathcal{B}} : \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n \mapsto (\alpha_1, \ldots, \alpha_n)^T$$

More general strategy for encoding linear maps as matrices:

For vector spaces U and V, with dimensions $\dim(U) = \ell$ and $\dim(V) = k$. . .

- Fix an ordered basis $\mathcal{A}$ of U, and use $\mathrm{Rep}_{\mathcal{A}} : U \to F^\ell$ to work with U concretely.
- Fix an ordered basis $\mathcal{B}$ of V, and use $\mathrm{Rep}_{\mathcal{B}} : V \to F^k$ to work with V concretely.
- Represent linear functions $f : U \to V$ as matrices $X_f \in M_{k,\ell}(F)$ by way of

$$\begin{tikzcd} V & F^k \arrow[l, "\mathrm{Rep}_{\mathcal{B}}^{-1}"'] \\ U \arrow[u, dotted, "f"] \arrow[r, "\mathrm{Rep}_{\mathcal{A}}"'] & F^\ell \arrow[u, "X_f"'] \end{tikzcd}$$

$X_f = \mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(f) \longleftarrow$ Notation

We call $\mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(f)$ the **matrix representation** of f

(with respect to the ordered bases $\mathcal{A}$ and $\mathcal{B}$).

Example

Consider the function

$$f : P_3(\mathbb{R}) \to M_2(\mathbb{R}) \quad \text{defined by} \quad p(x) \mapsto \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}.$$

[Polynomial evaluation is linear, so f is linear.] For example,

$$f : 2x - 5x^2 \mapsto \begin{pmatrix} [2(0) - 5(0)^2] & [2(1) - 5(1)^2] \\ [2(2) - 5(2)^2] & [2(3) - 5(3)^2] \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -16 & -39 \end{pmatrix}.$$

To represent f using a matrix, we'll need to fix ordered bases of $\mathcal{P}(\mathbb{R})$ and $M_2(\mathbb{R})$, respectively: let

$\mathcal{A} = \langle 1, x, x^2, x^3 \rangle,$ the standard ordered basis of $\mathcal{P}(\mathbb{R})$, and
$\mathcal{B} = \langle E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2} \rangle,$ the standard ordered basis of $M_2(\mathbb{R})$,

(recall that $E_{i,j}$ has a 1 in row i, column j, and 0's elsewhere).

Our job is to build $X_f = \mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(f)$ as follows:

(I) compute $f(\mathbf{a}_i)$ for each $\mathbf{a}_i \in \mathcal{A}$;
(II) encode the result as a linear combination of $\mathbf{b}_i$'s (compute $\mathrm{Rep}_{\mathcal{B}}(f(\mathbf{a}_i))$);
(III) set the ith column of X_f to be the result (set $\mathbf{col}_i = \mathrm{Rep}_{\mathcal{B}}(f(\mathbf{a}_i))$).

Consider the function

$$f : P_3(\mathbb{R}) \to M_2(\mathbb{R}) \quad \text{defined by} \quad p(x) \mapsto \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}.$$

Fix ordered bases

$\mathcal{A} = \langle 1, x, x^2, x^3 \rangle,$ the standard ordered basis of $\mathcal{P}(\mathbb{R})$, and
$\mathcal{B} = \langle E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2} \rangle,$ the standard ordered basis of $M_2(\mathbb{R})$,

(recall that $E_{i,j}$ has a 1 in row i, column j, and 0's elsewhere).

(I) We compute:	(II) which can be written as:
$1 \xmapsto{f} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$= 1E_{1,1} + 1E_{1,2} + 1E_{2,1} + 1E_{2,2}$
$x \xmapsto{f} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$	$= 0E_{1,1} + 1E_{1,2} + 2E_{2,1} + 3E_{2,2}$
$x^2 \xmapsto{f} \begin{pmatrix} 0^2 & 1^2 \\ 2^2 & 3^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 9 \end{pmatrix}$	$= 0E_{1,1} + 1E_{1,2} + 4E_{2,1} + 9E_{2,2}$
$x^3 \xmapsto{f} \begin{pmatrix} 0^3 & 1^3 \\ 2^3 & 3^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8 & 27 \end{pmatrix}$	$= 0E_{1,1} + 1E_{1,2} + 8E_{2,1} + 27E_{2,2}$

$f : P_3(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $p(x) \mapsto \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}$.

(I) We compute:

$$1 \overset{f}{\longmapsto} 1E_{1,1} + 1E_{1,2} + 1E_{2,1} + 1E_{2,2} \xmapsto{\mathrm{Rep}_{\mathcal{B}}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \leftarrow \text{(II) compute } \mathrm{Rep}_{\mathcal{B}}(f(\mathbf{a}_i))$$

$$x \overset{f}{\longmapsto} 0E_{1,1} + 1E_{1,2} + 2E_{2,1} + 3E_{2,2} \xmapsto{\mathrm{Rep}_{\mathcal{B}}} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$x^2 \overset{f}{\longmapsto} 0E_{1,1} + 1E_{1,2} + 4E_{2,1} + 9E_{2,2} \xmapsto{\mathrm{Rep}_{\mathcal{B}}} \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix}$$

$$x^3 \overset{f}{\longmapsto} 0E_{1,1} + 1E_{1,2} + 8E_{2,1} + 27E_{2,2} \xmapsto{\mathrm{Rep}_{\mathcal{B}}} \begin{pmatrix} 0 \\ 1 \\ 8 \\ 27 \end{pmatrix}$$

(III) Inserting the resulting vectors as columns, we get

$$X_f = \mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}.$$

Returning to our example, we saw

$$f : 2x - 5x^2 \mapsto \begin{pmatrix} 0 & -3 \\ -16 & -39 \end{pmatrix}.$$

If we want to use our matrix representation $\mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}$ instead, we

(I) convert $2x - 5x^2$ into a vector in F^4:

$$2x - 5x^2 = 0 * 1 + 2 * x + (-5) * x^2 + 0 * x^3 \xmapsto{\mathrm{Rep}_{\mathcal{A}}} \begin{pmatrix} 0 \\ 2 \\ -5 \\ 0 \end{pmatrix};$$

(II) Compute the matrix $\mathrm{Rep}_{\mathcal{A}}^{\mathcal{B}}(f)$ applied to the vector $\mathrm{Rep}_{\mathcal{A}}(2x - 5x^2)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 * 0 + 0 * 2 + 0 * (-5) + 0 * 0 \\ 1 * 0 + 1 * 2 + 1 * (-5) + 1 * 0 \\ 1 * 0 + 2 * 2 + 4 * (-5) + 8 * 0 \\ 1 * 0 + 3 * 2 + 9 * (-5) + 27 * 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -16 \\ -39 \end{pmatrix}.$$

(III) Convert the result back into an element of $M_2(\mathbb{R})$ by way of $\mathrm{Rep}_{\mathcal{B}}^{-1}$:

$$\begin{pmatrix} 0 \\ -3 \\ -16 \\ -39 \end{pmatrix} \xmapsto{\mathrm{Rep}_{\mathcal{B}}^{-1}} 0E_{1,1} + (-3)E_{1,2} + (-16)E_{2,1} + (-39)E_{2,2} = \begin{pmatrix} 0 & -3 \\ -16 & -39 \end{pmatrix} \checkmark$$