Lecture 10:

Linear extension
Rank/Nullity Theorem
Isomorphisms

Linear extensions: concrete constructions of linear maps

Question. Are there any linear functions $h : \mathbb{R}^2 \to \mathbb{R}^3$ that sends

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}3\\2\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}-1\\1\\5\end{pmatrix}? \qquad (*)$$

Answer. For any $(x, y) \in \mathbb{R}^2$, we know

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}.$$

Hence

$$\begin{aligned} h\left(\begin{pmatrix}x\\y\end{pmatrix}\right) &= h\left(x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right) = h\left(x\begin{pmatrix}1\\0\end{pmatrix}\right) + h\left(y\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= xh\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + yh\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = x\begin{pmatrix}3\\2\\0\end{pmatrix} + y\begin{pmatrix}-1\\1\\5\end{pmatrix} \\ &= \begin{pmatrix}x\cdot 3\\x\cdot 2\\x\cdot 0\end{pmatrix} + \begin{pmatrix}y\cdot(-1)\\y\cdot 1\\y\cdot 5\end{pmatrix} = \begin{pmatrix}x\cdot 3 + y\cdot(-1)\\x\cdot 2 + y\cdot 1\\x\cdot 0 + y\cdot 5\end{pmatrix}. \end{aligned}$$

So yes! There's a *unique* linear function that satisfies $(*)$.

Any linear transformation is determined by the image of a basis of the domain!

Thm. Let U and V be vector spaces over a field F, and let B be a basis of U. For each $\mathbf{b} \in B$, fix some $\mathbf{v_b} \in V$. Then there exists a unique linear transformation $h : U \to V$ that satisfies

$$h(\mathbf{b}) = \mathbf{v_b} \quad \text{for each } \mathbf{b} \in B.$$

In particular, for any $\mathbf{u} \in U$, there's a "unique" way to write $\mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, where $c_i \in F, \mathbf{b}_i \in B$. Then we define

$$h(\mathbf{u}) = c_1\mathbf{v}_{\mathbf{b}_1} + \cdots + c_n\mathbf{v}_{\mathbf{b}_n}.$$

Writing $h : B \to V$ defined by $H : \mathbf{b} \mapsto \mathbf{v_b}$, we say H extends linearly to $h : U \to V$, or that h is a linear extension of H.

See Book (Ch. Two, §II, Thm. 1.9) for proof.

Sketch: We have to check each of the following.

1. **Existence.** Check that the function above is
   (a) Well-defined: The image is in the codomain (follows from closure) and is independent of representatives (doesn't depend on how you write $\mathbf{u}$ as a linear combination over B).
   (b) Linear: similar to our examples, check that $h(\mathbf{u} + \mathbf{u}') = h(\mathbf{u}) + h(\mathbf{u}')$ and $h(\lambda\mathbf{u}) = \lambda h(\mathbf{u})$.
   (c) Does what it says it does: $h(\mathbf{b}) = \mathbf{v_b}$ for all $\mathbf{b} \in B$.
2. **Uniqueness.**
   If $g : U \to V$ *also* satisfies $g(\mathbf{b}) = \mathbf{v_b}$ for all $\mathbf{b} \in B$, then $g = h$.

Any linear transformation is determined by the image of a basis of the domain!

Thm. Let U and V be vector spaces over a field F, and let B be a basis of U. For each $\mathbf{b} \in B$, fix some $\mathbf{v_b} \in V$. Then there exists a unique linear transformation $h : U \to V$ that satisfies $h(\mathbf{b}) = \mathbf{v_b}$ for each $\mathbf{b} \in B$.

In particular, for any $\mathbf{u} \in U$, there's a "unique" way to write $\mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, where $c_i \in F, \mathbf{b}_i \in B$. Then we define $h(\mathbf{u}) = c_1\mathbf{v}_{\mathbf{b}_1} + \cdots + c_n\mathbf{v}_{\mathbf{b}_n}$.

Writing $H : B \to V$ defined by $H : \mathbf{b} \mapsto \mathbf{v_b}$, we say H extends linearly to $h : U \to V$, or that h is a linear extension of H.

Note. This theorem says something **very powerful**:
Given vector spaces U and V over F, and a basis B of U, the linear functions

$$\{h : U \to V \mid h \text{ is linear }\}$$

are in bijection with functions

$$\{H : B \to V\}.$$

Every linear map $h : U \to V$ *restricts uniquely* to a function $H : B \to V$; and every function $H : B \to V$ *extends uniquely* to a linear map $h : U \to V$.

Next week: Use this fact to encode linear functions as matrices.

Caution!

Things can go wrong when we try to do this with a set that is not a basis!

Exercise: Try to extend the function

$$H : (1,0) \mapsto (1,1), \quad (0,1) \mapsto (0,2), \quad \text{and} \quad (1,1) \mapsto (3,-1)$$

to a linear function $h : \mathbb{R}^2 \to \mathbb{R}^2$. What goes wrong?

Rank and nullity

Recall from last time: Let U, V be vector spaces over a field F, and let $h : U \to V$ be a linear function (a.k.a. homomorphism).

The range space of h is

$$\mathcal{R}(h) = h(U) = \{h(\mathbf{u}) \mid \mathbf{u} \in U\};$$

and the null space of f is

$$\mathcal{N}(h) = h^{-1}(\mathbf{0}_V) = \{\mathbf{u} \in U \mid h(\mathbf{u}) = \mathbf{0}_V\}.$$

Both are vector spaces (prove using subspace criterion), and hence we can talk about their dimensions.
In particular, the rank of h is $\mathrm{rank}(h) = \dim(\mathcal{R}(h))$; and
the nullity of h is $\mathrm{nullity}(h) = \dim(\mathcal{N}(h))$.

Example. Last time, we considered $h : \mathbb{R}^5 \to \mathbb{R}^2$ defined by $(s, t, x, y, z) \mapsto (4x, x - y)$, we computed that

$$\mathcal{R}(f) = \mathbb{R}^2 \quad \text{and} \quad \mathcal{N}(f) = \{(s, t, 0, 0, z) \mid s, t, z \in \mathbb{R}\} = \mathbb{R}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}.$$

So $\mathrm{rank}(f) = 2$ and $\mathrm{nullity}(f) = 3$. [*Notice that* $2 + 3 = \dim(\mathbb{R}^5)$]

Rank-nullity theorem

Theorem. Let U, V be vector spaces over a field F, and let $h : U \to V$ be a linear function. Then

$$\dim(U) = \mathrm{nullity}(h) + \mathrm{rank}(h).$$

Proof. Let $\mathcal{A}$ be a basis of $\mathcal{N}(\mathsf{h})$. In particular, $\mathcal{A}$ is a linearly independent subset of U, and hence there is some basis $\mathcal{X}$ of U that contains $\mathcal{A}$.

[Lecture 7: *Every independent set extends to a basis*].

Let $\mathcal{B} = \mathcal{X} - \mathcal{A}$, so that $\mathcal{X} = \mathcal{A} \sqcup \mathcal{B}$ (the disjoint union), and hence

$$|\mathcal{X}| = |\mathcal{A}| + |\mathcal{B}|$$

$\dim(U)$ → $|\mathcal{X}|$

$\dim(\mathcal{N}(h)) = \mathrm{nullity}(H)$ → $|\mathcal{A}|$

$\dim(F\mathcal{B})$ (whatever $F\mathcal{B}$ is...) → $|\mathcal{B}|$

Goal: show $\dim(F\mathcal{B}) = \mathrm{rank}(h)$

We will show that

(1) $h(\mathcal{B})$ is in bijection with $\mathcal{B}$ (so that $|\mathcal{B}| = |h(\mathcal{B})|$); and

(2) $h(\mathcal{B})$ is a basis of $\mathcal{R}(h)$ (so that $\dim(\mathcal{R}(h)) = |h(\mathcal{B})|$).

Hence, we will be able to conclude that

$$\mathrm{rank}(h) = \dim(\mathcal{R}(h)) = |h(\mathcal{B})| = |\mathcal{B}|,$$

which will prove our theorem.

So far: Let U, V be vector spaces over a field F, and let $h : U \to V$ be a linear function. Let

- $\mathcal{A}$ be a basis of $\mathcal{N}(h) = \{\mathbf{u} \in U \mid h(\mathbf{u}) = \mathbf{0}\}$ (the *null space* of h);
- $\mathcal{X}$ be a basis of U that contains $\mathcal{A}$; and

  [guaranteed to exists because $\mathcal{A}$ is linearly independent]
- $\mathcal{B} = \mathcal{X} - \mathcal{A}$.

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(1) Show $|\mathcal{B}| = |h(\mathcal{B})|$. [Recall $h(\mathcal{B}) = \{h(\mathbf{b}) \mid \mathbf{b} \in \mathcal{B}\}$]

Specifically, we'll show that $h : \mathcal{B} \to h(\mathcal{B})$ is a bijection.
It's surjective by definition, so we really just need to check that it's injective!

Let $\mathbf{b}, \mathbf{b}' \in \mathcal{B}$, and suppose that $h(\mathbf{b}) = h(\mathbf{b}')$. Then

$$\mathbf{0} = h(\mathbf{b}) - h(\mathbf{b}') = h(\mathbf{b} - \mathbf{b}');$$

so that $\mathbf{b} - \mathbf{b}' \in \mathcal{N}(h)$. Expanding $\mathbf{b} - \mathbf{b}'$ in the basis $\mathcal{A}$ (of $\mathcal{N}(h)$),

$$\mathbf{b} - \mathbf{b}' = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n,$$

we can see that either $\mathbf{b} = \mathbf{b}'$ or (since $\mathbf{b}, \mathbf{b}' \in \mathcal{X} - \mathcal{A}$) we have a contradiction of $\mathcal{X}$'s linear independence! Hence $h : \mathcal{B} \to h(\mathcal{B})$ is injective.

$$\boxed{\text{Thus } |\mathcal{B}| = |h(\mathcal{B})|.} \checkmark$$

So far: Let U, V be vector spaces over a field F, and let $h : U \to V$ be a linear function. Let

- $\mathcal{A}$ be a basis of $\mathcal{N}(h) = \{\mathbf{u} \in U \mid h(\mathbf{u}) = \mathbf{0}\}$ (the *null space* of h);
- $\mathcal{X}$ be a basis of U that contains $\mathcal{A}$; and

  [guaranteed to exists because $\mathcal{A}$ is linearly independent]
- $\mathcal{B} = \mathcal{X} - \mathcal{A}$.

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(2) Show $h(\mathcal{B})$ is a basis of $\mathcal{R}(h) = \{h(\mathbf{u}) \mid \mathbf{u} \in U\}$.

Spanning: Let $\mathbf{v} \in \mathcal{R}(h)$ (*Goal:* show $\mathbf{v} \in \mathrm{span}(h(\mathcal{B}))$).
Let $\mathbf{u} \in h^{-1}(\mathbf{v})$, meaning that $h(\mathbf{u}) = \mathbf{v}$. Since $\mathbf{u} \in U$, we can expand it in the basis $\mathcal{X} = \mathcal{A} \sqcup \mathcal{B}$; writing

$$\mathbf{u} = c_1\mathbf{a}_1 + \cdots + c_k\mathbf{a}_k + d_1\mathbf{b}_1 + \cdots + d_\ell\mathbf{b}_\ell$$

for some $c_i, d_i \in F$, $\mathbf{a}_i \in \mathcal{A}$, and $\mathbf{b}_i \in \mathcal{B}$. But then

$$\begin{aligned}
\mathbf{v} &= h(\mathbf{u}) = h(c_1\mathbf{a}_1 + \cdots + c_k\mathbf{a}_k + d_1\mathbf{b}_1 + \cdots + d_\ell\mathbf{b}_\ell) \\
&= c_1 h(\mathbf{a}_1) + \cdots + c_k h(\mathbf{a}_k) + d_1 h(\mathbf{b}_1) + \cdots + d_\ell h(\mathbf{b}_\ell) && (h \text{ is linear}) \\
&= c_1\mathbf{0} + \cdots + c_k\mathbf{0} + d_1 h(\mathbf{b}_1) + \cdots + d_\ell h(\mathbf{b}_\ell) && (\mathbf{a}_i \in \mathcal{N}(h)) \\
&= d_1 h(\mathbf{b}_1) + \cdots + d_\ell h(\mathbf{b}_\ell) \in \mathrm{span}(h(\mathcal{B})). \checkmark
\end{aligned}$$

So far: Let U, V be vector spaces over a field F, and let $h : U \to V$ be a linear function. Let

- $\mathcal{A}$ be a basis of $\mathcal{N}(h) = \{\mathbf{u} \in U \mid h(\mathbf{u}) = 0\}$ (the *null space* of h);
- $\mathcal{X}$ be a basis of U that contains $\mathcal{A}$; and

  [guaranteed to exists because $\mathcal{A}$ is linearly independent]
- $\mathcal{B} = \mathcal{X} - \mathcal{A}$.

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(2) Show $h(\mathcal{B})$ is a basis of $\mathcal{R}(h) = \{h(\mathbf{u}) \mid \mathbf{u} \in U\}$.

Independent: [Similarly to showing that $|h(\mathcal{B})| = |\mathcal{B}|$, we'll see that $h(\mathcal{B})$ is independent because its preimage is independent *from itself* **and** *from* $\mathcal{A}$.]
Let $d_1, \ldots, d_\ell \in F$ and $h(\mathbf{b}_1), \ldots, h(\mathbf{b}_\ell) \in h(\mathcal{B})$ such that

$$\begin{aligned} \mathbf{0} &= d_1 h(\mathbf{b}_1) + \cdots + d_\ell h(\mathbf{b}_\ell) \\ &= h(d_1 \mathbf{b}_1 + \cdots + d_\ell \mathbf{b}_\ell). \qquad (h \text{ is linear}). \end{aligned}$$

Hence, $d_1 \mathbf{b}_1 + \cdots + d_\ell \mathbf{b}_\ell \in \mathcal{N}(h) = \mathrm{span}(\mathcal{A})$. But again, either

$$d_1 \mathbf{b}_1 + \cdots + d_\ell \mathbf{b}_\ell = \mathbf{0},$$

or we have a contradiction of $\mathcal{X}$ being linearly independent. And if it *is* $\mathbf{0}$, then by $\mathcal{B}$'s independence, we know $d_1 = \cdots = d_\ell = 0$.

Hence $h(\mathcal{B})$ is linearly independent. ✓ [Concluding our proof of the Theorem.]

You try. Define $h : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by linearly extending

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}2\\1\\1\end{pmatrix}.$$

1. What exactly is h? Compute $h\left(\begin{pmatrix}a\\b\\c\end{pmatrix}\right)$ (in terms of $a, b, c \in \mathbb{R}$).
2. Compute $\mathcal{N}(h)$.
3. Compute $\mathcal{R}(h)$.
4. Give a basis $\mathcal{A}$ of $\mathcal{N}(h)$ (there are lots of examples—pick the easiest one you can think of).
5. Give a basis $\mathcal{X}$ of $\mathbb{R}^3$ that contains $\mathcal{A}$ (there are lots of examples—pick the easiest one you can think of).
6. Let $\mathcal{B} = \mathcal{X} - \mathcal{A}$. For each $\mathbf{b} \in \mathcal{B}$, compute $h(\mathbf{b})$.
7. Verify that $|h(\mathcal{B})| = |\mathcal{B}|$ and that the set $h(\mathcal{B})$ is a basis for $\mathcal{R}(h)$.

Injective linear functions

Theorem. A linear function $h : U \to V$ is injective if and only if $\mathcal{N}(h) = 0$.

Corollary. If $h : U \to V$ is linear and V is finite-dimensional, then the following are equivalent:

1. h is injective;
2. $\text{nullity}(h) = 0$;
3. $\text{rank}(h) = \dim(\mathbf{U})$;
4. If $\mathcal{B}$ is a basis for V, then $h(\mathcal{B})$ is a basis for $\mathcal{R}(h)$.

Isomorphisms

We call a bijective linear function an isomorphism.

Note: For any $h : U \to V$, the function $h : U \to \mathcal{R}(h)$ is surjective by definition. So $h : U \to \mathcal{R}(h)$ is an isomorphism if and only if $\text{nullity}(h) = 0$.

Recall that a function $f : X \to Y$ is bijective if and only if it has a two-sided inverse, i.e. a function $g : Y \to X$ such that

$$f \circ g = \text{id}_Y \quad \textit{and} \quad g \circ f = \text{id}_X.$$

Lemma.
If $h : U \to V$ is an isomorphism, then $h^{-1} : V \to U$ is also an isomorphism.

Proof: Exercise.

Example. Given an ordered basis $B = \langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$ of a vector space V, the representation $\text{Rep}_B : V \to F^n$ is an isormophism.
For example, using the standard ordered bases, we have the isomorphisms

$$\mathcal{P}_n(F) \to F^{n+1} \quad \text{defined by} \quad c_0 + c_1 x + \cdots + c_n x^n \mapsto (c_0, c_1, \ldots, c_n);$$

and

$$M_2(F) \to F^4 \quad \text{defined by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d).$$

Isomorphisms

We say that U is isomorphic to V if there exists an isomorphism $h : U \to V$. If so, we write $U \cong V$.

Examples: We just saw that $P_n(F) \cong F^{n+1}$ and $M_2(F) \cong F^4$.

AMAZING Theorem. Suppose U and V are finite-dimensional vector spaces. Then $U \cong V$ if and only if $\dim(U) = \dim(V)$.

Cor. If V is a finite-dimensional vector space, and $h : V \to V$ is linear, then the following are equivalent:

1. h is injective;
2. h is surjective;
3. h is an isomorphism.

However, if V is infinite-dimensional, there are linear maps that are injective but not surjective, and vice versa.