Lecture 9:

Linear transformations, a.k.a. homomorphisms
Range, preimage, and nullspace
Linear extension

Reminder:

- Exam 1 goes out in class on Thursday.
- You're allowed one 8.5" $\times$11" sheet of notes.
- Homework 4 is due by 3pm on Thursday, but LaTeX is not required.

Last time: We say a function $f : U \to V$ is linear (or is a linear transformation) if it satisfies

$$f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1) + f(\mathbf{u}_2) \quad \text{and} \quad f(\lambda \mathbf{u}) = \lambda f(\mathbf{u})$$

for all $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in U$ and $\lambda \in F$.

General terms:	"structure-preserving map" "homomorphism"

We say f preserves addition and scaling—the structure that is intrinsic to what it means to be a vector space!

Example: Scaling is linear.

Let V be a vector space over a field F, and fix $\alpha \in F$. Then

$$\begin{aligned} f : \ & V \to V \\ & \mathbf{v} \mapsto \alpha \mathbf{v} \end{aligned}$$

is a linear transformation.

Check: for any $\mathbf{u}, \mathbf{v} \in V$, $\lambda \in F$,

$$f(\mathbf{u} + \mathbf{v}) =$$

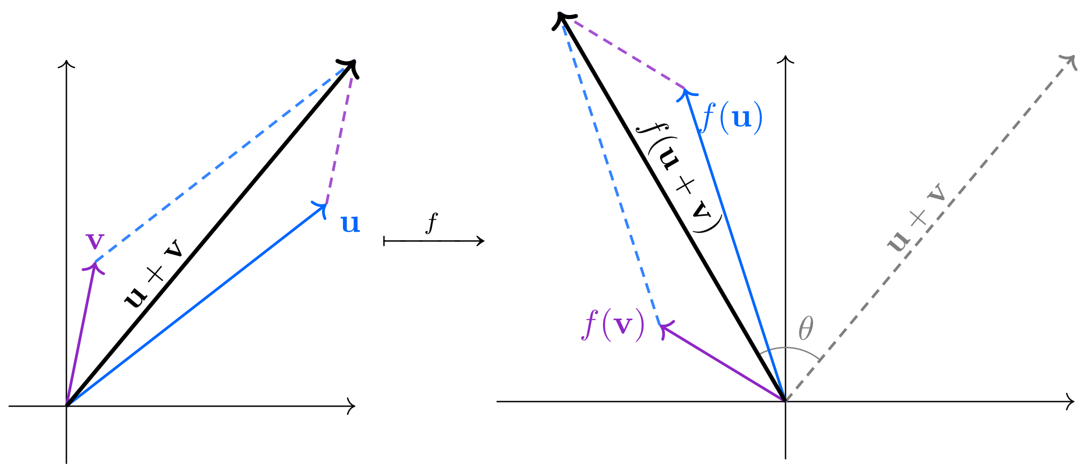
$$f(\lambda \mathbf{v}) =$$

Example. Rotation about the origin is linear.

Let $V = \mathbb{R}^2$ and fix $\theta \in \mathbb{R}$. Then rotation around the origin by θ,

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$$

is a linear transformation.



We really should check: for any $\mathbf{u}, \mathbf{v} \in V$, $\lambda \in F$,

$$f(\mathbf{u} + \mathbf{v}) =$$

$$f(\lambda\mathbf{v}) =$$

You try: Which of the following functions are linear?

1. $f : \mathbb{R} \to \mathbb{R}^2$ defined by $x \mapsto (0, 4x)$.
2. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto 3x - y$.
3. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto xy$.
4. $f : \mathbb{R}^2 \to M_2(\mathbb{R})$ defined by
$$(x, y) \mapsto \begin{pmatrix} x & x + y \\ 0 & 2y \end{pmatrix}.$$
5. $f : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by
$$p(x) \mapsto \frac{d}{dx}p(x).$$

Claim. Fix $a, b, c, d \in F$. Then the function

$$\begin{aligned} f: \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (ax + by, cx + dy) \end{aligned}$$

is linear.

Check:

$$\begin{aligned} f((x, y) + (x', y')) &= f((x + x', y + y')) \\ &= \big(a(x + x') + b(y + y'), c(x + x') + d(y + y')\big) \\ &= \big((ax + by) + (ax' + by'), (cx + yd) + (cx' + dy')\big) \\ &= \big(ax + by, cx + yd) + (ax' + by', cx' + dy') \\ &= f((x, y)) + f((x', y')); \quad \checkmark \end{aligned}$$

$$\begin{aligned} f(\lambda(x, y)) &= f((\lambda x, \lambda y)) \\ &= \big(a(\lambda x) + b(\lambda y), c(\lambda x) + d(\lambda y)\big) \\ &= \big(\lambda(ax + by), \lambda(cx + dy)\big) \\ &= \lambda(ax + by, cx + dy) \\ &= \lambda f((x, y)). \quad \checkmark \end{aligned}$$

Some properties of linear functions.

Let U, V be vector spaces over a field F, and let $f : U \to V$ be a linear function.

Lemma 1. Letting $\mathbf{0}_U$ and $\mathbf{0}_V$ denote the additive identities of U and V, respectively (as usual), we have

$$f(\mathbf{0}_U) = \mathbf{0}_V.$$

"Every linear map sends additive identity to additive identity."

Proof. (Consider $0_F \cdot \mathbf{u}$ for some $\mathbf{u} \in U$.)

Lemma 2. The range/image of f,

$$\begin{aligned} f(U) &= \{f(\mathbf{u}) \mid \mathbf{u} \in U\} \\ &= \{\mathbf{v} \in V \mid \text{ there is some } \mathbf{u} \in U \text{ such that } f(\mathbf{u}) = \mathbf{v}\} \end{aligned}$$

is a subspace of V.

Proof. Recall the subspace criterion: our goal is to show that

(1) $f(U) \neq \emptyset$, and

(2) for all $\mathbf{v}, \mathbf{w} \in f(U)$ and $\lambda \in F$, we have $\mathbf{v} + \lambda\mathbf{w} \in f(U)$.

The book calls $f(U)$ the range space of f, denoted $\mathcal{R}(f)$ (only curlier).

Let U, V be vector spaces over a field F, and let $f : U \to V$ be a linear function.

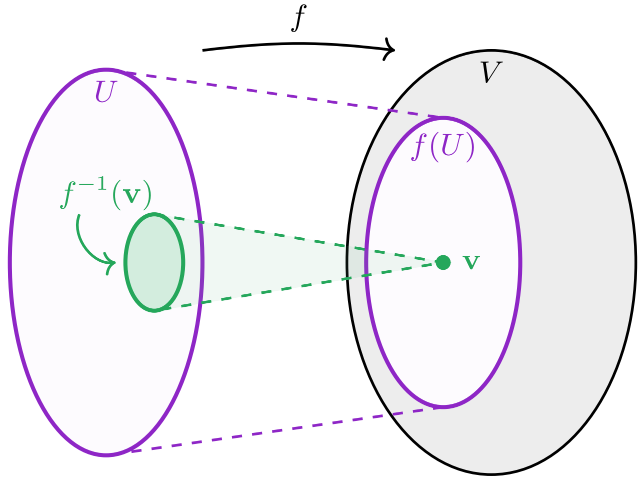
Lemma 1. $f(\mathbf{0}_U) = \mathbf{0}_V$.

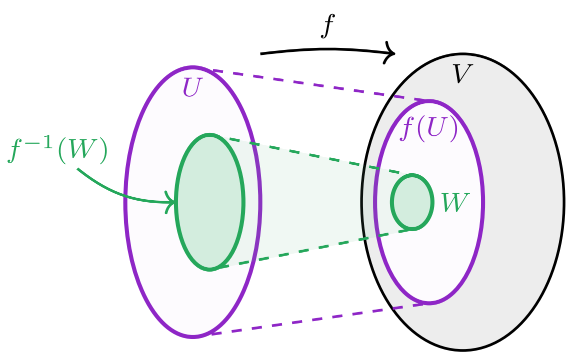
Lemma 2. The range/image of f, $f(U) = \{f(\mathbf{u}) \mid \mathbf{u} \in U\}$, is a subspace of V.

For any $\mathbf{v} \subseteq f(U)$, define the preimage ("inverse image") of $\mathbf{v}$ as

$$f^{-1}(\mathbf{v}) = \{\mathbf{u} \in U \mid f(\mathbf{u}) = \mathbf{v}\}.$$

Note: in general, we don't expect $f^{-1}(\mathbf{v})$ to be a single point unless f is injective!





For most $\mathbf{v} \in f(U)$, note that $f^{-1}(\mathbf{v})$ is *not* a subspace of U. However...

For a subspace $W \subseteq f(U)$, the preimage ("inverse image") of W is

$$f^{-1}(W) = \bigcup_{\mathbf{w} \in W} f^{-1}(\mathbf{w}) = \{\mathbf{u} \in U \mid f(\mathbf{u}) \in W\}.$$

Lemma 3. The preimage of a subspace $W \subseteq f(U)$ is a subspace of U.

Let U, V be vector spaces over a field F, and let $f : U \to V$ be a linear function.

Lemma 1. $f(\mathbf{0}_U) = \mathbf{0}_V$.

Lemma 2. The range/image of f, $f(U) = \{f(\mathbf{u}) \mid \mathbf{u} \in U\}$, is a subspace of V.

Lemma 3. The preimage of a subspace $W \subseteq f(U)$, given by

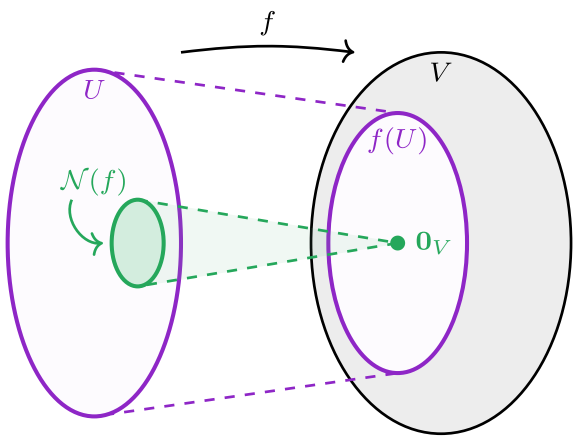
$$f^{-1}(W) = \{\mathbf{u} \in U \mid f(\mathbf{u}) \in W\},$$

is a subspace of U.

In particular, the kernel, or nullspace, of f is

$$\mathcal{N}(f) = f^{-1}(\mathbf{0}_V) = \{\mathbf{u} \in U \mid f(\mathbf{u}) = \mathbf{0}_V\}.$$

Since $0 = \{\mathbf{0}_V\}$ is a subspace of V and $f(\mathbf{0}_U) = \mathbf{0}_V$ (so that 0 is a subspace of $f(U)$), we know that $\mathcal{N}(f)$ is a subspace of U.



Example. Consider $f : \mathbb{R}^5 \to \mathbb{R}^2$ defined by $(s, t, x, y, z) \mapsto (4x, x - y)$.

Let's compute

1. $f(\mathbb{R}^5) = \{(a, b) \in \mathbb{R}^2 \mid a = 4x$ and $b = x - y$ for some $(s, t, x, y, z) \in \mathbb{R}^5\}$

2. $f^{-1}((4, -5)) = \{(s, t, x, y, z) \in \mathbb{R}^5 \mid 4x = 4$ and $x - y = -5\}$

3. $\mathcal{N}(f) = \{(s, t, x, y, z) \in \mathbb{R}^5 \mid 4x = 0$ and $x - y = 0\}$

Let U, V be vector spaces over a field F, and let $f : U \to V$ be a linear function (a.k.a. homomorphism).

The range space of f is
$$\mathcal{R}(f) = f(U) = \{f(\mathbf{u}) \mid \mathbf{u} \in U\};$$
and the null space of f is
$$\mathcal{N}(f) = f^{-1}(\mathbf{0}_V) = \{\mathbf{u} \in U \mid f(\mathbf{u}) = \mathbf{0}_V\}.$$

These are vector spaces, and so we are interested in their dimensions!

The rank of f is $\mathrm{rank}(f) = \dim(f(U))$; and the nullity of f is $\mathrm{nullity}(f) = \dim(\mathcal{N}(f))$.

Example. For $f : \mathbb{R}^5 \to \mathbb{R}^2$ defined by $(s, t, x, y, z) \mapsto (4x, x - y)$, we computed that
$$\mathcal{R}(f) = \mathbb{R}^2 \quad \text{and} \quad \mathcal{N}(f) = \{(s, t, 0, 0, z) \mid s, t, z \in \mathbb{R}\} = \mathbb{R}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}.$$
So $\mathrm{rank}(f) = 2$ and $\mathrm{nullity}(f) = 3$. [*Notice that* $2 + 3 = \dim(\mathbb{R}^5)$]

Example. For $f : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $p(x) \mapsto \frac{d}{dx}p(x)$, we have
$$a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_1 + 2a_2x + 3a_3x^2.$$
So
$$\mathcal{R}(f) = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \mid \deg(p(x)) \leqslant 2\} \text{“="} \mathcal{P}_2(x); \quad \text{and}$$
$$\mathcal{N}(f) = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \mid \tfrac{d}{dx}p(x) = 0\} = \{ \text{ constant polynomials } \} \text{“="} \mathbb{R}.$$
Hence $\mathrm{rank}(f) = 3$ and $\mathrm{nullity}(f) = 1$. [*Notice that* $3 + 1 = \dim(\mathcal{P}_3(\mathbb{R}))$]

Linear extensions: concrete constructions of linear maps

Question. Are there any linear functions $f : \mathbb{R}^2 \to \mathbb{R}^3$ that sends
$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}3\\2\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}-1\\1\\5\end{pmatrix}? \qquad (*)$$

Answer. For any $(x, y) \in \mathbb{R}^2$, we know
$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}.$$
Hence
$$\begin{aligned} f\left(\begin{pmatrix}x\\y\end{pmatrix}\right) &= f\left(x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right) = f\left(x\begin{pmatrix}1\\0\end{pmatrix}\right) + f\left(y\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= xf\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + yf\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = x\begin{pmatrix}3\\2\\0\end{pmatrix} + y\begin{pmatrix}-1\\1\\5\end{pmatrix} \\ &= \begin{pmatrix}x \cdot 3\\x \cdot 2\\x \cdot 0\end{pmatrix} + \begin{pmatrix}y \cdot (-1)\\y \cdot 1\\y \cdot 5\end{pmatrix} = \begin{pmatrix}x \cdot 3 + y \cdot (-1)\\x \cdot 2 + y \cdot 1\\x \cdot 0 + y \cdot 5\end{pmatrix}. \end{aligned}$$
So yes! There's a *unique* linear function that satisfies $(*)$.

Any linear transformation is determined by the image of a basis of the domain!

Thm. Let U and V be vector spaces over a field F, and let B be a basis of U. For each $\mathbf{b} \in B$, fix some $\mathbf{v_b} \in V$. Then there exists a unique linear transformation $f : U \to V$ that satisfies

$$f(\mathbf{b}) = \mathbf{v_b} \quad \text{for each } \mathbf{b} \in B.$$

In particular, for any $\mathbf{u} \in U$, there's a "unique" way to write $\mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, where $c_i \in F, \mathbf{b}_i \in B$. Then we define

$$f(\mathbf{u}) = c_1\mathbf{v}_{\mathbf{b}_1} + \cdots + c_n\mathbf{v}_{\mathbf{b}_n}.$$

Writing $F : B \to V$ defined by $F : \mathbf{b} \mapsto \mathbf{v_b}$, we say F extends linearly to $f : U \to V$, or that f is a linear extension of F.

See Book (Ch. Two, §II, Thm. 1.9) for proof.

Sketch: We have to check each of the following.

1. **Existence.** Check that the function above is
   (a) Well-defined: The image is in the codomain (follows from closure) and is independent of representatives (doesn't depend on how you write $\mathbf{u}$ as a linear combination over B).
   (b) Linear: similar to our examples, check that $f(\mathbf{u} + \mathbf{u}') = f(\mathbf{u}) + f(\mathbf{u}')$ and $f(\lambda\mathbf{u}) = \lambda f(\mathbf{u})$.
   (c) Does what it says it does: $f(\mathbf{b}) = \mathbf{v_b}$ for all $\mathbf{b} \in B$.
2. **Uniqueness.**
   If $g : U \to V$ *also* satisfies $g(\mathbf{b}) = \mathbf{v_b}$ for all $\mathbf{b} \in B$, then $g = f$.

Note. This theorem says something **very powerful**:
Given vector spaces U and V over F, and a basis B of U, the linear functions

$$\{f : U \to V \mid f \text{ is linear }\}$$

are in bijection with functions

$$\{F : B \to V\}.$$

Every linear map $f : U \to V$ *restricts uniquely* to a function $F : B \to V$; and every function $F : B \to V$ *extends uniquely* to a linear map $f : U \to V$.

Next time: Use this fact to encode linear functions as matrices.

Caution!

Things can go wrong when we try to do this with a set that is not a basis!

Exercise: Try to extend the function

$$F : (1, 0) \mapsto (1, 1), \quad (0, 1) \mapsto (0, 2), \quad \text{and} \quad (1, 1) \mapsto (3, -1)$$

to a linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$. What goes wrong?