Lecture 8:

Row/column spaces

Rank

Linear transformations (a beginning)

Warmup

Recall: A relation on a set $\mathcal{S}$ is a subset $\mathcal{X} \subseteq \mathcal{S} \times \mathcal{S}$ or pairs of elements. We write $s \sim t$ to mean $(s,t) \in \mathcal{X}$, and think of this statement as "s is related to t". A relation is an equivalence relation if it satisfies three conditions:

1. Reflexivity. For all $s \in \mathcal{S}$, we have $s \sim s$.
2. Symmetry. For any $s, t \in \mathcal{S}$ such that $s \sim t$, we must also have $t \sim s$.
3. Transitivity.
   For any $s, t, u \in \mathcal{S}$ such that $s \sim t$ and $t \sim u$, we must also have $s \sim u$.

[Fav. example: $=$ is an equivalence relation on any set.]

Fix $k, \ell \in \mathbb{Z}_{>0}$. On the set of matrices $M_{k,\ell}(F)$, define the relation

$A \sim B$ whenever "B can be reached by a sequence of row operations on A"

You check: verify that this defines an *equivalence* relation on $M_{k,\ell}(F)$.

Hint: For matrices A and B, writing the rows of A as $\mathbf{a}_1, \ldots, \mathbf{a}_k$ and the rows of B as $\mathbf{b}_1, \ldots, \mathbf{b}_k$, we have...

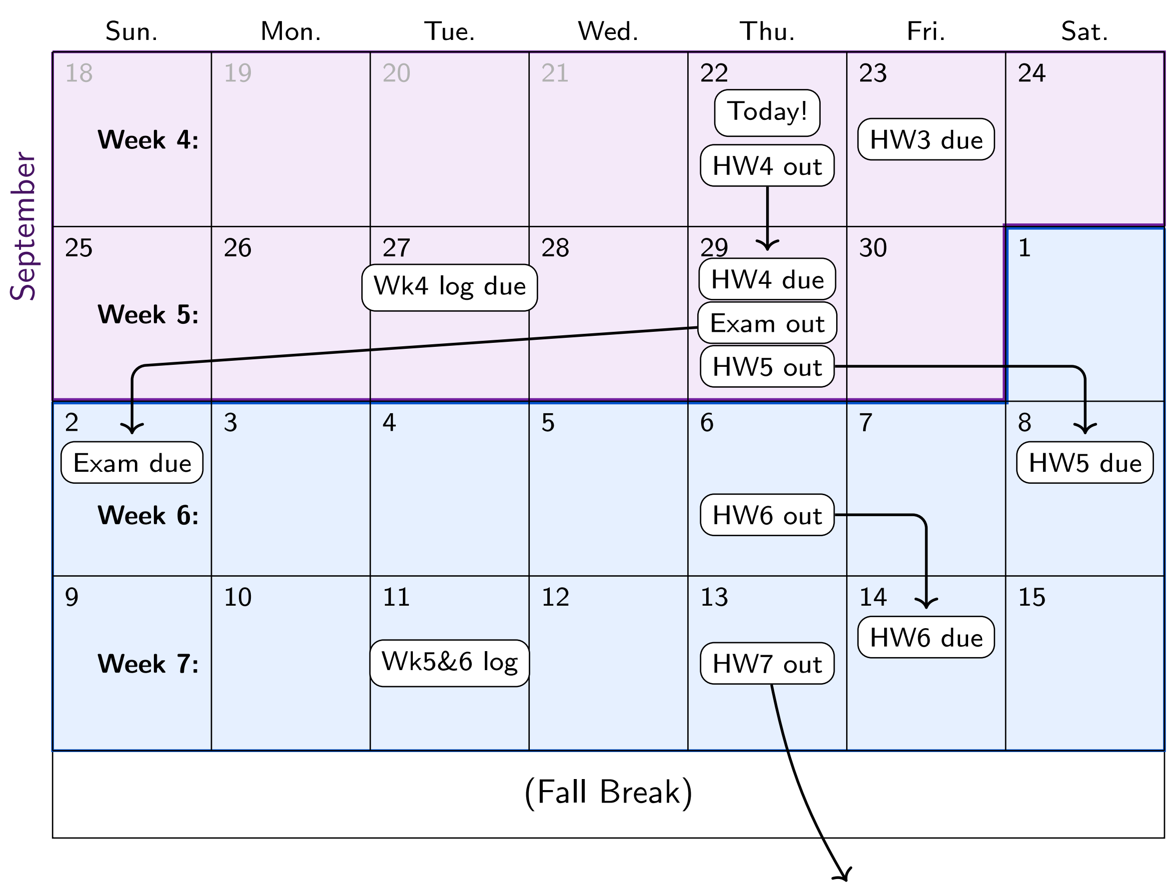
if $A \xmapsto{\mathbf{a}_i \leftrightarrow \mathbf{a}_j} B$, then $B \xmapsto{\mathbf{b}_i \leftrightarrow \mathbf{b}_j} A$;

if $A \xmapsto{\mathbf{a}_i \mapsto \alpha \mathbf{a}_i} B$ for $\alpha \neq 0$, then $B \xmapsto{\mathbf{b}_i \mapsto \frac{1}{\alpha} \mathbf{b}_i} A$; and

if $A \xmapsto{\mathbf{a}_i \mapsto \mathbf{a}_i + \alpha \mathbf{a}_j} B$ for $i \neq j$, then $B \xmapsto{\mathbf{b}_i \mapsto \mathbf{b}_i - \alpha \mathbf{b}_j} A$.

Some info about next week's exam and other work:

- Exam handed out in class, Thursday 9/29;
  not to be opened before 3pm that day;
  due on Gradescope Sunday 10/2.
- Time limit: 3 hours in one sitting.
  - Test written to be doable in 1 hour.
  - Time limit does not include time spent scanning/uploading exam.
  - Do not open the exam until you're ready to begin.
- Closed book/notes/internet/other people's brains/etc.
  except for one 8.5" × 11" sheet of notes.
- Covers Weeks 1–4 (Lectures 1–8, HW 1–4, Chapters One and Two).
- **Homework 4**:
  - Due THURSDAY 9/29 by 3pm.
  - LaTeX not required.
- **Weekly logs**:
  - Week 4 due next Tuesday as usual.
  - No log due Tuesday 10/4 (immediately after exam)
  - Week 5 and Week 6 will be combined into one assignment (due 10/11).



Last time: Dimension is a rigid statistic on vector spaces! If V be a finite-dimensional v.s./F, and $W \subseteq V$ is a subspace, then

$$\dim(W) \leqslant \dim(V) \qquad \text{and} \qquad \dim(W) < \dim(V) \text{ if and only if } W \subsetneq V.$$

In particular, linearly independent sets have size bounded above by n, spanning sets have size bounded below by n, and sets of exactly size n are either bases or they fail at *both spanning and independence*.

We also saw that in finite-dimensional spaces,

- every independent set extends to a basis (is contained in a basis), and
- every spanning set contains a basis.

Example: Prove that the following is a basis of $\mathbb{R}^4$:

$$B = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\}.$$

Since $|B| = 4 = \dim(\mathbb{R}^4)$, we know that it's independent if and only if it is a spanning set, so I only need to test for one of those. But the equation

$$c_1 \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + c_3 \begin{pmatrix}1\\1\\1\\0\end{pmatrix} + c_4 \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad \text{has aug. matrix} \quad \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

which is already in row echelon form. So I can already see that it will have unique solution (and $c_i = 0$ is a solution, so I know that's the one).

Last time: Given a matrix $A \in M_{m,n}(F)$,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix},$$

we associate the set of column vectors

$$\left\{ \begin{pmatrix}a_{1,1}\\ \vdots\\ a_{m,1}\end{pmatrix}, \begin{pmatrix}a_{1,2}\\ \vdots\\ a_{m,2}\end{pmatrix}, \dots, \begin{pmatrix}a_{1,n}\\ \vdots\\ a_{m,n}\end{pmatrix} \right\}.$$

Define the pivot columns of A as those column vectors corresponding to the pivots of A's reduced form. [Compute the reduced form of A to know *where* to look; but read the columns off of A, *not* its reduced form.]

Lemma. The pivot columns of A form a basis for $\mathrm{ColSpace}(A)$.

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Example. Find a basis for

$$V = F\left\{\begin{pmatrix}1\\-1\\3\\0\end{pmatrix},\begin{pmatrix}7\\-9\\5\\-4\end{pmatrix},\begin{pmatrix}-2\\3\\2\\2\end{pmatrix},\begin{pmatrix}-1\\1\\-3\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\\0\end{pmatrix}\right\} \subseteq \mathbb{R}^4.$$

The associated matrix is

$$A = \begin{pmatrix} \overset{(1)}{1} & \overset{(2)}{7} & -2 & -1 & \overset{(5)}{0} & 0 \\ -1 & -9 & 3 & 1 & 0 & 1 \\ 3 & 5 & 2 & -3 & 1 & 2 \\ 0 & -4 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad \text{which reduces to} \quad \begin{pmatrix} \overset{(1)}{1} & \overset{(2)}{0} & \frac{3}{2} & -1 & \overset{(5)}{0} & -1 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the pivots are in columns 1, 2, and 5, we know that $V = \mathrm{ColSpace}(A)$ has basis

$$B = \left\{\begin{pmatrix}1\\-1\\3\\0\end{pmatrix},\begin{pmatrix}7\\-9\\5\\-4\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}.$$

Is there a "better" basis?

For example, $B' = \left\{\begin{pmatrix}1\\0\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$ is also a basis for V. How???

Let $A \in M_{k,\ell}(F)$ be a $k \times \ell$ matrix. (k rows, ℓ columns)

- The column vectors are the vectors $\mathcal{C} = \left\{\begin{pmatrix}a_{1,j}\\ \vdots \\ a_{k,j}\end{pmatrix} \middle| j = 1, \dots, \ell\right\}$.
- The row vectors are the vectors $\mathcal{R} = \{(a_{i,1} \ \cdots \ a_{i,\ell}) \mid i = 1, \dots, k\}$.
- The column space of A is $\mathrm{ColSpace}(A) = F\mathcal{C}$,
  the span of the column vectors of A.
- The row space of A is $\mathrm{RowSpace}(A) = F\mathcal{R}$,
  the span of the row vectors of A.
- The column rank of A is $\dim(\mathrm{ColSpace}(A))$
- The row rank of A is $\dim(\mathrm{RowSpace}(A))$.

So far: We have a nice understanding of $\mathrm{ColSpace}(A)$ in terms of A's reduced form E:

- $\mathrm{ColSpace}(A)$ has a basis consisting of the pivot columns of A (those marked by pivots in E); so that
- the column rank of A is the number of pivots in E.

We'd like:

- a similar description of $\mathrm{RowSpace}(A)$, and
- a reason to care.

Back to row operations!

Let's focus in on row vectors by writing

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,\ell} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,\ell} \\ \vdots & \vdots & & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,\ell} \end{pmatrix} \quad \text{as} \quad \begin{pmatrix} \text{---} & \mathbf{a}_1 & \text{---} \\ \text{---} & \mathbf{a}_2 & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_k & \text{---} \end{pmatrix}.$$

Consider how the row space of a matrix is affected by row operations:

1. Row swapping.

$$A = \begin{pmatrix} & \vdots & \\ \text{---} & \mathbf{a}_i & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_j & \text{---} \\ & \vdots & \end{pmatrix} \xrightarrow{\mathbf{a}_i \leftrightarrow \mathbf{a}_j} \begin{pmatrix} & \vdots & \\ \text{---} & \mathbf{a}_j & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_i & \text{---} \\ & \vdots & \end{pmatrix} = B.$$

The set of **row** vectors has not changed, so

$$\text{RowSpace}(A) = \text{RowSpace}(B).$$

Starting with matrix A...

1. Row swapping. Swap rows i and j to get B.

   The set of row vectors has not changed, so

$$\text{RowSpace}(A) = \text{RowSpace}(B).$$

2. Row scaling. For some $\alpha \neq 0$, replace $\mathbf{a}_i$ with $\alpha\mathbf{a}_i$ to get B.

$$A = \begin{pmatrix} & \vdots & \\ \text{---} & \mathbf{a}_i & \text{---} \\ & \vdots & \end{pmatrix} \xrightarrow{\mathbf{a}_i \mapsto \alpha\mathbf{a}_i} \begin{pmatrix} & \vdots & \\ \text{---} & \alpha\mathbf{a}_i & \text{---} \\ & \vdots & \end{pmatrix} = B.$$

   The row vectors of B are linear combinations of the row vectors of A, so

$$\text{RowSpace}(A) \supseteq \text{RowSpace}(B).$$

3. Row combinations. For some $i \neq j$, replace $\mathbf{a}_i$ with $\mathbf{a}_i + \alpha\mathbf{a}_j$ to get B.

$$A = \begin{pmatrix} & \vdots & \\ \text{---} & \mathbf{a}_i & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_j & \text{---} \\ & \vdots & \end{pmatrix} \xrightarrow{\mathbf{a}_i \mapsto \mathbf{a}_i + \alpha\mathbf{a}_j} \begin{pmatrix} & \vdots & \\ \text{---} & \mathbf{a}_i + \alpha\mathbf{a}_j & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_j & \text{---} \\ & \vdots & \end{pmatrix} = B.$$

   The row vectors of B are linear combinations of the row vectors of A, so

$$\text{RowSpace}(A) \supseteq \text{RowSpace}(B).$$

Starting with matrix A and applying row operations to get B...
In all three cases, $\text{RowSpace}(A) \supseteq \text{RowSpace}(B)$.

But all three row operations are reversible! (see warmup)
So $\text{RowSpace}(A) \subseteq \text{RowSpace}(B)$ by the same arguments.

Hence $\boxed{\text{RowSpace}(A) = \text{RowSpace}(B)}$.

Ex. The matrix

$$\begin{pmatrix} 1 & -1 & 3 & 0 \\ 7 & -9 & 5 & -4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{reduces to} \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So

$$\mathbb{R}\left\{\begin{pmatrix} 1 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -9 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\} = \mathbb{R}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

[From our example at the beginning]

Caution!
We put the first set of vectors into the matrix as *rows* and then reduced!! (Up until this moment, we have been putting vectors into columns of matrices.)

Back to equivalence relations...

Given an equivalence relation $\sim$ on a set $\mathcal{S}$, and an element $s \in \mathcal{S}$, the set

$$[s] = \{t \in \mathcal{S} \mid t \sim s\}$$

is called the equivalence class of s.

The equivalence classes have a really nice property (put three ways):

- If $t \in [s]$, then $[s] = [t]$ (we call the elements of $[s]$ representatives of $[s]$).
- For any $s, t \in \mathcal{S}$, either $[s] = [t]$ or $[s] \cap [t] = \varnothing$.
- The equivalence classes partition $\mathcal{S}$ (meaning they break $\mathcal{S}$ into disjoint subsets).

In the warmup, we showed that row operations give us a nice equivalence relation on $M_{k,\ell}(F)$ given by sequences of row operations.
Pushing that language a little further...

1. In each equivalence class, there is a *unique* element that's in reduced row echelon form; this is typically our favorite representative.
2. Solution sets (to the associated homogeneous system of equations) are constant on equivalence classes.

Now:

3. Row spaces are constant on equivalence classes.
4. Row rank (dim. of the row space) is constant on equivalence classes.

Given a matrix A and its reduced echelon form E...

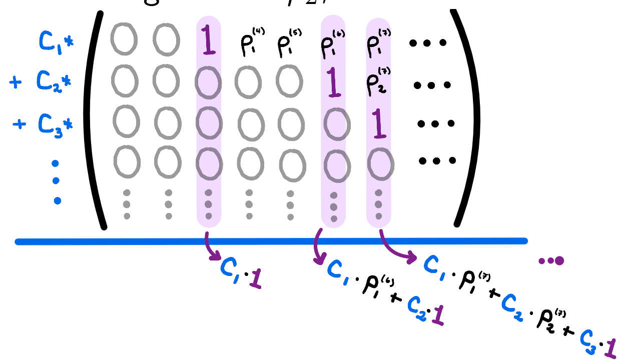
So far:

1. $\mathrm{RowSpace}(A) = \mathrm{RowSpace}(E)$
2. $\mathrm{RowRank}(A) = \mathrm{RowRank}(E)$
3. The pivot columns of A form a basis of $\mathrm{ColSpace}(A)$.
4. $\mathrm{ColRank}(A) = \#$ pivots of E.

Lemma. The non-zero row vectors of any row echelon form B of A form a linearly independent set, and hence form a basis for $\mathrm{RowSpace}(A)$. In particular, the non-zero row vectors of E form a basis of $\mathrm{RowSpace}(A)$.

Pf (sketch). Denote the row vectors of B by $\rho_1, \ldots, \rho_m$, and consider solutions to $\mathbf{0} = c_1\rho_1 + \cdots + c_m\rho_m$.

Compare the coefficients corresponding to the leading term of ρ_1; then those corresponding to the leading term of ρ_2; and so on...



Given a matrix A and its reduced echelon form E...

(Amazing) Corollary. $\mathrm{RowRank}(A) = \mathrm{ColRank}(A)$.

Pf. By the lem, the dimension of the column space of A is equal to the number of pivots of E (one pivot per non-zero row), which is also the dimension of the row space of A.

Define the rank of a matrix A as

$$\mathrm{rk}(A) = \mathrm{RowRank}(A) = \mathrm{ColRank}(A) = \#\text{pivots in } E.$$

Putting it all together:

1. $\mathrm{ColSpace}(A)$ has basis {pivot columns *of* A}.

   [This basis *is* a subset of your original collection of vectors.]

   Problem: Given a set S of vectors, find a subset of S that's a basis for FS.

   Sol'n: Insert S as cols, row reduce to find pivot locations, and take corresp. vectors from S.

2. $\mathrm{RowSpace}(A)$ has basis {non-zero rows of E}.

   [*Caution:* This basis is almost never a subset of your original collection of vectors.]

   Problem: Given a set S of vectors, find a basis of FS that's as close to $\mathcal{E}$ as possible.

   Sol. Insert S as rows, row reduce, take the resulting row vectors.

3. The rank of A is a statistic relating to *both* the row space *and* the column space of A (and of any other matrix related by row operations).
4. The *solution space* of the homogeneous system associated to A has dimension

$$\#\{\text{ free variables }\} = \#\{\text{ columns of } A \} - \mathrm{rk}(A)$$

Next time: Functions between vector spaces.

Let U and V be vector spaces over the same field F.

Q. What properties would we *want* out of a function $f : U \to V$?

Illustrative example: Given an ordered basis B of U, we have a function

$$\mathrm{Rep}_B : U \to F^n.$$

It's *meaningful* because

1. it preserves $+$; $\quad \mathrm{Rep}_B(\mathbf{u}_1 + \mathbf{u}_2) = \mathrm{Rep}_B(\mathbf{u}_1) + \mathrm{Rep}_B(\mathbf{u}_2)$
2. it preserves scaling; $\quad \mathrm{Rep}_B(\alpha \cdot \mathbf{u}) = \alpha \cdot \mathrm{Rep}_B(\mathbf{u})$
3. and it's bijective.

Really, we like 1 and 2 because they mean Rep_B "respects" the vector space structure (otherwise, what's the point of noticing U is a vector space in the first place??). We like 3 because it means we can move back and forth between U and F^n faithfully, without losing any information (too strong!).

We say a function $f : U \to V$ is linear (or is a linear transformation) if it satisfies

$$f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1) + f(\mathbf{u}_2) \quad \text{and} \quad f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$$

for all $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in U$ and $\alpha \in F$.

General terms:	"structure-preserving map" "homomorphism"