Lecture 7 Warmup

Let F be a field (with $0 \neq 1$) and let V be a vector space over F. Last time, we proved the *Exchange Lemma* (Lemma 1 below); on Homework 3, you will also prove the following Lemma 2.

Lemma 1 (Exchange Lemma)**.** *Let B be a basis of V and let $\mathbf{v} \in V - \{\mathbf{0}\}$. Then there exists an element $\mathbf{b} \in B$ such that $B' = (B - \{\mathbf{b}\}) \cup \{\mathbf{v}\}$ is also a basis of V. Specifically, $\mathbf{b}$ is any vector that appears with non-zero coefficient in the expansion of $\mathbf{v}$ in B.*

Lemma 2. *Let $S \subseteq T \subseteq V$ be subsets. If S generates V and T is linearly independent, then $S = T$.*

Warmup: Work in groups of 2–3. Returning to the big theorem we sketched a proof of at the end of Lecture 6 (restated a little here to make it more straightforward to prove), we'll walk through the proof more carefully now. For each of the marked (*) statements and steps in the following proof of Theorem 3, fill out the details. In other words, convince yourself/each other of their truth and/or suss out *why* each statement is true or *why* each step is possible.

Theorem 3. *Suppose V has at least one basis of size n. Then any other basis also has size n.*

Proof. Let A and B be bases of V, and assume $|A| = n$. If $A \subseteq B$, then by Lemma 2, $A = B$ (1). Hence $|B| = |A| = n$.

Otherwise (if $A \nsubseteq B$), let $B_1 = B$, and take $\mathbf{a}_1 \in A - B_1$. Since $\mathbf{a}_1 \neq \mathbf{0}$ (2), when we write $\mathbf{a}_1$ as a linear combination of elements of B_1, there's some $\mathbf{b}_1 \in B_1 - A$ (3) that appears with non-zero coefficient. So by Lemma 1, the set $B_2 = (B_1 - \{\mathbf{b}_1\}) \cup \{\mathbf{a}_1\}$ is also a basis of V.

If $A \subseteq B_2$, then $A = B_2$. Hence $|B| = |B_2|$ (4) $= |A| = n$. Otherwise, take some $\mathbf{a}_2 \in A - B_2$ and expand $\mathbf{a}_2$ in the basis B_2.

Since $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent (5), when we write $\mathbf{a}_2$ as a linear combination of elements of B_2, there is some $\mathbf{b}_2 \in B_2 - A$ (6) that contributes (meaning that it has non-zero coefficient in the expansion of $\mathbf{a}_2$), and hence $B_3 = (B_2 - \{\mathbf{b}_2\}) \cup \{\mathbf{a}_2\}$ is a basis (again by the Exchange Lemma).

Continuing to recurse, at each step we'll take $\mathbf{a}_k \in A - B_k$ and expand it in the basis B_k. Then since $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is linearly independent (7), when we write $\mathbf{a}_k$ as a linear combination of elements of B_k, there must be some $\mathbf{b}_k \in B_k - A$ that contributes with non-zero coefficient; and hence $B_{k+1} = (B_k - \{\mathbf{b}_k\}) \cup \{\mathbf{a}_k\}$ is a basis.

Since A is finite (with n elements), this process will end after at most n steps. (8) If i is the step where it ends (meaning $A \subseteq B_i$), then we can conclude

$$|B| = |B_1| = |B_2| = \cdots = |B_i| \ ^{(9)} = |A| = n,$$

as desired. ∎

Last questions:

(10) Where and why did it become important that I started moving elements of A into B, rather than the other way around?

(11) When did we actually use the fact that A was linearly independent? that B was linearly independent?

(12) When did we actually use the fact that A spans V? that B spans V?

Lecture 7:

More relationships between

independent sets, bases, and spanning sets

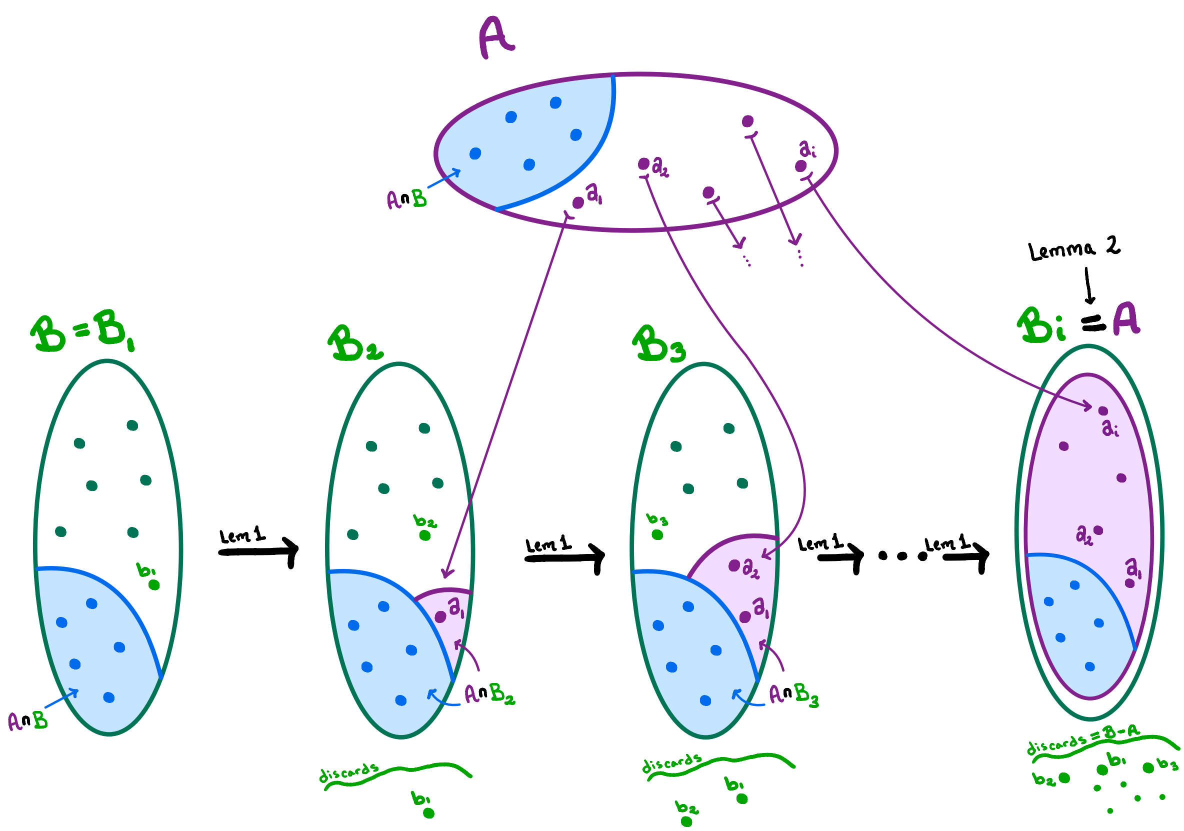
"Dimension is strict"

Column space of a matrix

Begin with Warmup

Unless otherwise stated:
Assume F is a field with more than one element (so that $0 \neq 1$).
Let V be a vector space over F.

Proof of Theorem 3



We can actually push the same proof of Theorem 3 to get much stronger characterization of dimension!

Theorem 4

Let V be a finite-dimensional v.s./F, with $\dim(V) = n$. Then

1. *every linearly independent set has at most n elements;*
2. *every spanning set has at least n elements; and*
3. *if $S \subseteq V$ has exactly n elements, then*

$$S \text{ spans } V \qquad \text{if and only} \qquad S \text{ is linearly independent.}$$

In particular, if $W \subseteq V$ is a subspace, then

$$\dim(W) \leqslant \dim(V) \qquad \text{and} \qquad \dim(W) < \dim(V) \text{ if and only if } W \subsetneq V.$$

Why would we care?

Claim: $\{(1,2),(3,4)\}$ is a basis for $\mathbb{C}^2$ (over $F = \mathbb{C}$).

To prove this last week, we would have had to...

1. For $x, y \in \mathbb{C}$, find $c_1, c_2 \in \mathbb{C}$ such that $(x, y) = c_1(1,2) + c_2(3,4)$.
2. Show that c_1, c_2 are unique (at least in the case where $x = y = 0$).

Now, we know that since $\dim_{\mathbb{C}}(\mathbb{C}^2) = 2 = |\{(1,2),(3,4)\}|$, then we know that $\{(1,2),(3,4)\}$ spans if and only if it's independent.

[*Independent:* $c_1(1,2) + c_2(3,4) = (0,0)$ implies either

$$c_1 = c_2 = 0 \qquad \text{or} \qquad (3,4) = c(1,2) \text{ for some } c \in \mathbb{C};$$

the latter is not possible.]

Thm. 4 Let V be a finite-dimensional v.s./F, with $\dim(V) = n$.
Then — Let $S \subseteq V$.

1. every linearly independent set has at most n elements; — "If S indep., then $|S| \leqslant n$."
2. every spanning set has at least n elements; and — "If $FS = V$, then $|S| \geqslant n$."
3. if $S \subseteq V$ has exactly n elements, then

$$S \text{ spans } V \qquad \text{if and only} \qquad S \text{ is linearly independent.}$$

In particular, if $W \subseteq V$ is a subspace, then

$$\dim(W) \leqslant \dim(V) \qquad \text{and} \qquad \dim(W) < \dim(V) \text{ if and only if } W \subsetneq V.$$

Proof. Let $S \subseteq V$, and let B be a basis of V.

For 1, by the warmup, we can iteratively move S into B using the Exchange Lemma (same-ish proof as that of Theorem 3), implying that $|S| \leqslant |B| = n$.

For 2, use the Exchange Lemma to iteratively move B into S, implying that $|S| \geqslant |B| = n$.

For 3, suppose $|S| = n$.

If S spans V, use the techniques of Lecture 5 to find a linearly independent subset $T \subseteq S$ that has the same span as S.

[Set up $c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0}$, row reduce, and discard $\mathbf{s}_i$'s corresponding to free var's.]

If S is independent, use the Exchange Lemma to move S into B. But since $S \subseteq B_i$ (the basis at the last step) and $|S| = |B_i| = |B|$, we must have $S = B_i$, and is therefore a basis. $\square$

Cor. Let V be a finite-dimensional v.s.$/F$, with $\dim(V) = n$. Let $S \subseteq V$.

1. If S is linearly independent, then S is contained in a basis.

   "Every independent set extends to a basis."

2. If S spans V, then S contains a basis.

   "Every spanning set contains a basis."

Proof.
1. Take a basis B. Feed S into B to get a basis B_i containing S.
2. Take one element of S at a time, iteratively building up an independent subset:

$$S_1 = \{\mathbf{s}_1\}, \quad S_2 = \{\mathbf{s}_1, \mathbf{s}_2\}, \quad \ldots .$$

If $i < n$, then S_i is not spanning (by Thm. 4), so there is some $v \in V - FS_i$. Since S *does* span, there's some element $\mathbf{s}_{i+1} \in S - S_i$ that contributes to expanding v in S. Iterate until $i = n$. By Thm. 4, since S_n is independent and $|S_n| = n$, it must be a basis.

Row operations revisited

What was that technique from Lecture 5 again? i.e. how did we use an aug. matrix and row ops to shrink a (finite) set to an independent subset with the same span.

We started with

$$S = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

and endeavored to solve

$$c_1 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + c_5 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To do so, we reduced

$$\left(\begin{array}{ccccc|c} 2 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \quad \text{to get} \quad \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 3/2 & 0 \\ 0 & 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)\ldots$$

Our observations:

1. Considering the solutions $c_3 = 1$ & $c_5 = 0$ and $c_3 = 0$ & $c_5 = 1$ showed $\mathbf{s}_3$ and $\mathbf{s}_5$ were in the span of $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_4\}$; and setting $c_3 = c_5 = 0$ forced $c_1 = c_2 = c_4 = 0$, showing that $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_4\}$ is independent.
2. The last column (of 0's) didn't have any effect (except to find FS).

Going the other way, given a matrix $A \in M_{m,n}(F)$,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix},$$

we associate a homogeneous system

$$c_1 \begin{pmatrix} a_{1,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + c_2 \begin{pmatrix} a_{1,2} \\ \vdots \\ a_{m,2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Notice that we're really thinking about the columns in A as individual vectors in F^n, inserted into one big array: we call these the column vectors of A. Define the pivot columns of A as those column vectors corresponding to the pivots of A's reduced form. [Compute the reduced form of A to know *where* to look; but read the columns off of A, *not* its reduced form.]

Ex. We saw $\begin{pmatrix} 2 & 0 & 2 & 0 & 3 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ reduces to $\begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 3/2 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}$,

so the pivot columns of A are $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

The whole point of our example (Lecture 5/last slide) was that the pivot columns of A have the same span as *all* of the column vectors of A.

The column space of A is the span of its column vectors:

$$\text{If } A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}, \quad \text{then} \quad \text{ColSpace}(A) = FS$$

$$\text{where} \quad S = \left\{ \begin{pmatrix} a_{1,1} \\ \vdots \\ a_{m,1} \end{pmatrix}, \begin{pmatrix} a_{1,2} \\ \vdots \\ a_{m,2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{pmatrix} \right\}.$$

Lemma. The pivot columns of A form a basis for $\text{ColSpace}(A)$.

Proof. (A formalization of our earlier example—for your reading only)

Let $S = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be the column vectors of A; so the homogeneous equation associated to A is

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}. \qquad (*)$$

Now, let $P = \{\mathbf{p}_1, \dots, \mathbf{p}_\ell\}$ be the pivot columns of A (so there are some $i_1 < \cdots < i_\ell$ such that $\mathbf{p}_1 = \mathbf{s}_{i_1}, \dots, \mathbf{p}_\ell = \mathbf{s}_{i_\ell}$)...

In our example where $\begin{pmatrix} 2 & 0 & 2 & 0 & 3 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ reduced to $\begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 3/2 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}$,

we have $\ell = 3$ (the number of pivots), and are setting $i_1 = 1$, $i_2 = 2$, and $i_3 = 4$, so that

$$\mathbf{p}_1 = \mathbf{a}_1, \quad \mathbf{p}_2 = \mathbf{a}_2, \quad \text{and} \quad \mathbf{p}_3 = \mathbf{a}_4.$$

The free variables in the associated homogeneous equation are

$$c_j \quad \text{for} \quad j \in \{1,2,3,4,5\} - \{1,2,4\} = \{3,5\}.$$

Lemma. The pivot columns of A form a basis for $\text{ColSpace}(A)$.

Proof. (A formalization of our earlier example—for your reading only)

Let $S = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be the column vectors of A; so the homogeneous equation associated to A is

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}. \qquad (*)$$

Now, let $P = \{\mathbf{p}_1, \dots, \mathbf{p}_\ell\}$ be the pivot columns of A (so there are some $i_1 < \cdots < i_\ell$ such that $\mathbf{p}_1 = \mathbf{s}_{i_1}, \dots, \mathbf{p}_\ell = \mathbf{s}_{i_\ell}$). Testing dependence of P is the same as testing for a solution to $(*)$ *that also has* all coefficients except $c_{i_1}, \dots c_{i_\ell}$ set to 0 (for all $j \in \{1, \dots, n\} - \{i_1, \dots, i_\ell\}$, we want $c_j = 0$). But the solution to the homogeneous system associated to E is exactly the same as that associated to A (the point of row operations). And by design, the column vectors of E in columns $i_1, \dots, i_\ell$ are the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_\ell$, which are independent. So $c_{i_1} = \cdots = c_{i_\ell} = 0$. Thus $\{\mathbf{p}_1, \dots, \mathbf{p}_\ell\}$ is independent.

"Row operations preserve (in)dependence of columns"

Now, suppose $\mathbf{v} \in \text{ColSpace}$. Then there is a solution to

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{v}. \qquad (\star)$$

But that's exactly the same as the solution space to the reduced form of $(A|\mathbf{v})$. In *that* solution, I find out that the coefficients c_j for $j \in \{1, \dots, n\} - \{i_1, \dots, i_\ell\}$ are all free; so I might as well set the free variables all to be 0. This means that there exists a solution to my equation $(\star)$ of the form $c_{i_1}\mathbf{p}_1 + \cdots + c_{i_\ell}\mathbf{p}_\ell = \mathbf{v}$; and hence $\mathbf{v} \in FP$. Hence $FS \subseteq FP$, and therefore $FS = FP$.

Exercises

Book: Ch Two, §III.3 #22: Give a basis for the column space of

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{pmatrix}.$$

Book: Ch Two, §III.2 #25: Give an argument showing that the following is a basis of $\mathbb{R}^4$ *without* doing any calculations (arithmetic).

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Generalization of the last problem:

A square matrix $A \in M_{n,n}(F)$ is said to be upper triangular if $a_{i,j} = 0$ whenever $i > j$: ($a_{i,j}$ for $i \leqslant j$ may or may not be 0)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n,n} \end{pmatrix}.$$

Argue that that column vectors of an upper-triangular matrix form a basis for F^n if and only if $a_{i,i} \neq 0$ for all $i = 1, \ldots, n$.