Lecture 6:
Bases
Dimension

Unless otherwise stated:
Assume F is a field with more than one element (so that $0 \neq 1$).
Let V be a vector space over F.

Last time

Let V be a vector space (over F), and let $S \subseteq V$ be a subset of V. There are two main ways we think about the span of S:

- Bottom-up: FS is the vector space that you can build out of S.
  Focus on S and see where you can get.
- Top-down: S is enough to build FS.
  Focus on the vector space FS and find a way to build it.

A set $S \subseteq V$ is *linearly independent* if
$$c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0} \quad \text{implies} \quad c_1 = \cdots = c_n = 0$$
for any $\mathbf{s}_1, \ldots, \mathbf{s}_n \in S$. Otherwise, if there exist $c_1, \ldots, c_n$ with at least some $c_\ell \neq 0$ such that $c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0}$, we say S is *linearly dependent*.

Other characterizations of linear (in)dependence:

- S is *independent* if and only if
  $$c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = d_1\mathbf{s}_1 + \cdots + d_n\mathbf{s}_n \quad \text{implies} \quad c_i = d_i \text{ for } i = 1, \ldots, n;$$
  i.e. each element in FS has a *unique* expression as a linear combination of elements of S.
- S is *dependent* if and only if
  $$\text{there exists } \mathbf{v} \in S \quad \text{such that} \quad \mathbf{v} \in F(S - \{\mathbf{v}\});$$
  i.e. there's redundancy in S as a set of building blocks for FS.

Bases

Still: Let V be a vector space over F.

A spanning set for V is a subset $S \subseteq V$ such that $FS = V$.

Think: S is "enough" to build/generate V.

A basis for V is a linearly independent spanning set.

Think: S is *minimal* in being "enough" to build/generate V.

Example: We showed last time that $\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Example: Another basis of $\mathbb{R}^3$ is the natural or standard basis

$$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad \text{where} \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Reflection questions (from last time):

1. We have lots of examples of vector spaces now. Can you come up with a basis for each of them? For example, can you find a basis for... (next slide)
   - $V = \mathcal{P}_n(\mathbb{R}) = \{f \in \mathbb{R}[x] \mid \deg(f) \leqslant n\}$ over $F = \mathbb{R}$;
   - $V = M_{2,2}(\mathbb{R}) = \{$ 2 × 2 matrices w/ coefs in $\mathbb{R}$ $\}$, over $F = \mathbb{R}$;
   - $V = 0 = \{\mathbf{0}\}$, the trivial vector space over a field F;
2. Many vector spaces have more than one basis. Under what circumstances will a basis of V be unique?
3. Does every vector space even *have* a basis? (How could you prove or disprove?)

Standard bases

There are usually lots of bases of a given vector space, and there will be very concrete examples where "non-standard" bases are important. But generally, we start to think about most of our favorite vector spaces in terms of natural choices of bases already.

The standard basis for...

- $V = F^n$ (over F) is

$$\mathcal{E}\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \quad \text{where} \quad \mathbf{e}_i = (0, \dots, 0, \underset{i\text{th coordinate}}{1}, 0, \dots, 0);$$

- $V = M_{a,b}(F)$ (over F) is $\mathcal{E} = \{E_{i,j} \mid i = 1, \dots a, j = 1, \dots, b\}$ where

$$E_{i,j} = \begin{array}{c} \begin{array}{ccccccc} & & & j\text{th col} & & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ i\text{th row} \\ \\ \\ \\ \end{array}$$

(0's everywhere except in the ith row and jth col);

- $V = F[x]$ (over F) is $\mathcal{E} = \{1, x, x^2, \dots\} = \{x^\ell \mid \ell \in \mathbb{Z}_{\geqslant 0}\}$;

  and $V = \mathcal{P}_n(F)$ (over F) is $\mathcal{E} = \{1, x, x^2, \dots, x^n\}$.

Putting together what we learned about span, together with the fact that a basis generates all of V we know...

Prop. Let V be a vector space over F, and let B be a basis of V. Then every element $\mathbf{v} \in V$ has a *unique* expression in the form $\mathbf{v} = \sum_{\mathbf{b} \in B} c_{\mathbf{b}} \mathbf{b}$;

i.e. there is a unique way to express $\mathbf{v}$ as a linear combination of elements of B (being smart about order and 0's).

[For example,

$$2\begin{pmatrix}1\\0\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad 5\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \text{and} \quad 5\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix} + 2\begin{pmatrix}1\\0\\0\end{pmatrix}$$

all count as the "same" linear combination.]

An ordered basis B is a basis together with a fixed order on its elements.

Note: This is what the book just calls a "basis"; this *not* standard.

We'll write

$$B = \langle \mathbf{v}_1, \dots, \mathbf{v}_n \rangle$$

to mean the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ considered in *that order*.

Aside: There is no standard notation for ordered bases—the literature uses ( ), [ ], etc. *Most* people just use { }, even though technically sets don't have order, but it can be helpful to have separate notation while we're first learning, so we'll use special notation for now.

An ordered basis B is a basis together with a fixed order on its elements). We'll write

$$B = \langle \mathbf{v}_1, \dots, \mathbf{v}_n \rangle$$

to mean the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ considered in *that order*.

For now, let's focus on cases where V has a *finite* basis.

yes: F^n, $M_{a,b}(F)$, $\mathcal{P}_n(F) = \{p \in F[x] \mid \deg(p) \leqslant n\}$ no: $F[x]$

Examples:

$\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ is the standard ordered basis of F^n;

$\langle E_{1,1}, E_{1,2}, \dots, E_{1,b}, E_{2,1} \dots, E_{2,b}, \dots, E_{a,b} \rangle$ is the standard ordered basis of $M_{a,b}(F)$;

$\langle 1, x, x^2, \dots, x^n \rangle$ is the standard ordered basis of $\mathcal{P}_n(F)$.

Example:

$$\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle, \quad \langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2 \rangle, \quad \text{and} \quad \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 \rangle$$

are all different ordered bases of $\mathbb{R}^3$ (corresponding to the same underlying basis).

Example. Let's consider the three ordered bases in $\mathbb{C}^3$ given by

$$\mathcal{E} = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle \qquad \text{with } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

$$A = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle \qquad \text{with } \mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{a}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \text{ and}$$

$$B = \langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \rangle \qquad \text{with } \mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

[*How do I know they're bases?* One can check that these are each independent sets using the techniques of last time, and the following confirms that they're spanning sets as well.]

We can solve relevant systems of linear equations (e.g. $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$) to find

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= x\ \mathbf{e}_1 + y\ \mathbf{e}_2 + z\ \mathbf{e}_3, \\ &= y\ \mathbf{a}_1 + z\ \mathbf{a}_2 + x\ \mathbf{a}_3, \qquad \text{and} \\ &= (x - y)\ \mathbf{b}_3 + (y - z)\ \mathbf{b}_2 + z\ \mathbf{b}_3. \end{aligned}$$

Encoding V using an ordered basis

Let $B = \langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$ be an ordered basis of a vector space V (over the field F). For each $\mathbf{v} \in V$, we again note that the expression

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

is *unique*, meaning that there is a bijection

$$\begin{array}{ccc} F^n & \longrightarrow & V \\ (c_1, \ldots, c_n) & \longmapsto & c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n. \end{array} \qquad (*)$$

> *Recall:* A map $f : X \to Y$ is well-defined if for all $x \in X$ we have
>
> (1) $f(x)$ is uniquely identified and (2) $f(x) \in Y$.
>
> A bijection, or bijective function, is a function $f : X \to Y$ that is both injective, meaning
>
> $$f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2,$$
>
> and surjective, meaning
>
> for all $y \in Y$, there is some $x \in X$ such that $f(x) = y$.

The function in $(*)$ is *well-defined* because B is ordered;
it is *surjective* because B is a spanning set; and
it is *injective* because B is independent.

In other words, the definition of ordered basis is *exactly* what's needed for $(*)$ to be well-defined and bijective.

Encoding V using an ordered basis

Let $B = \langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$ be an ordered basis of a vector space V (over F).
Recall that a function is bijective if and only if it's invertible;
so take $(*)$ from the last and turn it around (invert it) to get the bijection

$$\begin{array}{rccc} \mathrm{Rep}_B : & V & \longrightarrow & F^n \\ & c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n & \longmapsto & (c_1, \ldots, c_n). \end{array}$$

We call $c_1, \ldots, c_n$ the coordinates of

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

with respect to B, and

$$\mathrm{Rep}_B(\mathbf{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

is the representation of $\mathbf{v}$ with respect to B.

Example: Using the ordered bases $\mathcal{E}$, A, and B of $V = \mathbb{C}$ from before, we have

$$\mathrm{Rep}_{\mathcal{E}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathrm{Rep}_A\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ z \\ x \end{pmatrix}, \quad \text{and} \quad \mathrm{Rep}_B\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x-y \\ y-z \\ z \end{pmatrix}.$$

You try.

1. Consider the ordered basis $B = \langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \rangle$ of $\mathbb{Q}^3$, where

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

   (You *could* verify that B is independent by solving
   $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{0}$ for $c_1, c_2, c_3 \in \mathbb{Q}$.)

   Compute $\mathrm{Rep}_B(\mathbf{u})$ for

   (i) $\mathbf{u} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ and (ii) $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (where $x, y, z \in \mathbb{Q}$).

   [*Hint.* Start by solving $\mathbf{u} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$ for c_1, c_2, c_3.]

2. Let V be a vector space with ordered basis $B = \langle \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle$, so that

   $$\mathrm{Rep}_B(\mathbf{u}) = (c_1, \ldots, c_n) \quad \text{means } \mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

   Verify that, for any $\mathbf{u}, \mathbf{v} \in V$ and $\alpha \in F$, we have

$$\boxed{\mathrm{Rep}_B(\mathbf{u}+\mathbf{v}) = \mathrm{Rep}_B(\mathbf{u}) + \mathrm{Rep}_B(\mathbf{v}) \quad \text{and} \quad \mathrm{Rep}_B(\alpha\mathbf{u}) = \alpha\,\mathrm{Rep}_B(\mathbf{u}).}$$

   [*Hint.* Start the first identity by writing $\mathbf{v} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n$ and computing (I) $\mathrm{Rep}_B(\mathbf{u})$ and $\mathrm{Rep}_B(\mathbf{v})$, so to compute $\mathrm{Rep}_B(\mathbf{u}) + \mathrm{Rep}_B(\mathbf{v})$ using vector addition in F^n; and (II) $\mathbf{u}+\mathbf{v}$ (collecting like terms) and using *that* to compute $\mathrm{Rep}_B(\mathbf{u}+\mathbf{v})$.]

Dimension

Theorem

If V has bases B and C, then $|B| = |C|$.

(Proof in the finite case in a moment...)

In particular, if V has a basis B, then $|B|$ is a statistic for V, *not just* B.

[*Aside:* This is a statement about vector spaces with infinite bases as well, where $|B| = |C|$ means that there exists a bijection $B \to C$.]

Definition. We call the size of a basis B of V the dimension of V, denoted

$$\dim(V) = |B|.$$

If we need to emphasize what field we're working with, we can write $\dim_F(V)$.

Examples.

- $\dim(F^n) = n$ because $|\{\mathbf{e}_1, \dots, \mathbf{e}_n\}| = n$;
- $\dim(M_{a,b}(F)) = ab$ because $|\{E_{i,j} \mid i = 1, \dots, a, j = 1, \dots, b\}| = ab$;
- $\dim(\mathcal{P}_n(F)) = n + 1$ because $|\{1, x, \dots, x^n\}| = n + 1$;
- $\dim_{\mathbb{R}}(\mathbb{C}) = 2$ because $\{1, i\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$.

Thm. (Finite case)
If V has finite bases B and C, then $|B| = |C|$.

Lemma. (Exchange Lemma)
Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a (finite*) basis of V and let $\mathbf{v} \in V - \{\mathbf{0}\}$. Then there exists an element $\mathbf{b} \in B$ such that $B' = (B - \{\mathbf{b}\}) \cup \{\mathbf{v}\}$ is also a basis of V. (* also true in infinite case.)

Proof. Fix $\mathbf{v} \in V - \{\mathbf{0}\}$. Since B is a basis of V and $\mathbf{v} \neq 0$, we can write

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

for some $c_i \in F$ not all 0. Take ℓ such that $c_\ell \neq 0$ and solve for $\mathbf{b}_\ell$:

$$\mathbf{b}_\ell = \frac{1}{c_\ell}\mathbf{v} + \sum_{\substack{i=1,\dots n\\ i\neq\ell}} \left(\frac{-c_i}{c_\ell}\right)\mathbf{b}_i \in FB', \text{ where } B' = (B - \{\mathbf{b}_\ell\}) \cup \{\mathbf{v}\}. \qquad (\diamond)$$

To see that B' spans V, we see that for any $\mathbf{u} \in V$, we have

$$\begin{aligned}
\mathbf{u} &= d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n = d_\ell\mathbf{b}_\ell + \sum_{\substack{i=1,\dots n\\ i\neq\ell}} d_i\mathbf{b}_i && \text{for some } d_i \in F,\\
&= d_\ell\left(\frac{1}{c_\ell}\mathbf{v} + \sum_{\substack{i=1,\dots n\\ i\neq\ell}} \left(\frac{-c_i}{c_\ell}\right)\mathbf{b}_i\right) + \sum_{\substack{i=1,\dots n\\ i\neq\ell}} d_i\mathbf{b}_i && \text{by } (\diamond)\\
&= \frac{d_\ell}{c_\ell}\mathbf{v} + \sum_{\substack{i=1,\dots n\\ i\neq\ell}} \left(\frac{-d_\ell c_i}{c_\ell} + d_i\right)\mathbf{b}_i \in FB'.
\end{aligned}$$

$\boxed{\text{So } FB' = V.}$

(This shows $FB' \supseteq V$, but $FB' \subseteq V$ already by closure)

Lemma. (Exchange Lemma)
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a (finite) basis of V and let $\mathbf{v} \in V - \{\mathbf{0}\}$. Then there exists an element $\mathbf{b} \in B$ such that $B' = (B - \{\mathbf{b}\}) \cup \{\mathbf{v}\}$ is also a basis of V.

Proof (continued). We fixed $\mathbf{v} \in V - \{\mathbf{0}\}$, wrote $\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, and took ℓ such that $c_\ell \neq 0$. Then we defined $\boxed{B' = (B - \{\mathbf{b}_\ell\}) \cup \{\mathbf{v}\}}$ and showed $\boxed{FB' = V}$. It remains to show that B' is independent.
To that end, suppose (for some $\alpha, \alpha_i \in F$) we have

$$\mathbf{0} = \alpha\mathbf{v} + \sum_{\substack{i=1,\ldots,n\\ i\neq\ell}} \alpha_i\mathbf{b}_i$$

(how we always start to test for dependence)

$$= \alpha\left(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n\right) + \sum_{\substack{i=1,\ldots,n\\ i\neq\ell}} \alpha_i\mathbf{b}_i,$$

using our formula for $\mathbf{v}$,

$$= \alpha c_\ell\mathbf{b}_\ell + \sum_{\substack{i=1,\ldots,n\\ i\neq\ell}} (\alpha c_i + \alpha_i)\mathbf{b}_i,$$

combining like terms.

But B is a basis (and therefore is independent), so this implies that

$$\alpha c_\ell = 0 \qquad \text{and} \qquad \alpha c_i + \alpha_i = 0 \text{ for all } i \neq \ell.$$

We assumed $c_\ell \neq 0$, and hence $\boxed{\alpha = 0}$; and this further implies that $\boxed{\alpha_i = 0 \text{ for all } i \neq \ell}$ (as desired).
So B' is independent, and is therefore a basis. □

Lemma. (Exchange Lemma)
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a (finite*) basis of V and let $\mathbf{v} \in V - \{\mathbf{0}\}$. Then there exists an element $\mathbf{b} \in B$ such that $B' = (B - \{\mathbf{b}\}) \cup \{\mathbf{v}\}$ is also a basis of V.

Note: Our proof *more specifically* showed...
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a (finite) basis of V and let $\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \in V - \{\mathbf{0}\}$. Then **for any ℓ such that $c_\ell \neq 0$, we have that $B' = (B - \{\mathbf{b}_\ell\}) \cup \{\mathbf{v}\}$ is also a basis of V.**

(We cooked up a specific recipe for finding *what* to replace with $\mathbf{v}$.)

Theorem (Ch Two, §III, Thm. 2.4)

If V has finite bases B and C, then $|B| = |C|$.
Namely, dimension is well-defined.

Proof (sketch). [See book for full deatails]
Inductively move from $B \to C$, replacing one term at a time.

Step 0: Let $B_1 = B$.

Step i: If $C \subseteq B_i$, you're done. Otherwise...

- Take some element $\mathbf{v}_i \in C - B$, and find $\mathbf{u}_i \in B_0 - C$ such that the coefficient of $\mathbf{u}_i$ in $\mathrm{Rep}_{B_i}(\mathbf{v}_i)$ is not 0.
- Let $B_{i+1} = (B_i - \{\mathbf{u}_i\}) \cup \{\mathbf{v}_i\}$.

(Recurse step i until $C \subseteq B_i$.)

Homework: Show that if $C \subseteq B_i$ with C spanning V and B_i being indepenent, then $C = B_i$. □

Epilog: Some tips for translating between lecture and the book.

- The book defines a basis as an *ordered* set (that spans and is independent)—i.e. what we're calling an *ordered* basis. This isn't standard, so we'll differentiate between the two.
- The book notationally distinguishes between a linearly independent spanning set

  $$B = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\}; \qquad \text{(set)}$$

  and an *ordered* linearly independent spanning set (the thing they call basis)

  $$B = \langle\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\rangle. \qquad \text{(ordered set)}$$

  In LaTeX, those angle brackets *aren't* just < and >; the angle bracket symbols are more shallow (e.g. $<$ versus $\langle$). They're coded as `\langle` and `\rangle` ("left angle" and "right angle"). You also have the shortcuts \< and \> coded in our preambles for those two commands, respectively.

  Note: There is no standard convention for what to use for notation in the second case. If you're looking at other resources, keep an eye out for ( ) (thinking of an ordered basis as a sequence) or [ ]. But mostly, folks just use { }, even though technically sets don't have order, and use words to specify that they've fixed an order.