Lecture 5:
Linear independence
Bases

Unless otherwise stated:
Assume F is a field with more than one element (so that $0 \neq 1$).
Let V be a vector space over F.

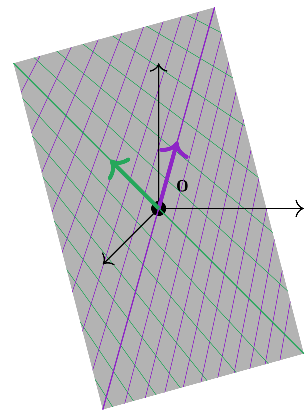
Last time: Let V be a vector space (over F), and let $S \subseteq V$ be a subset of V. The linear closure or span of S is the set

$$FS = \{a_1\mathbf{s}_1 + \cdots + a_n\mathbf{s}_n \mid n \in \mathbb{Z}_{\geqslant 0}, a_i \in F, \mathbf{s}_i \in S\}.$$

We also denote FS by $\text{span}_F(S)$ (book: [S]).

Back to our parameterized planes picture...

$$H = \{a\mathbf{u} + b\mathbf{v} \mid a, b \in \mathbb{R}\}:$$



Lemma. (Two.I.2.15)
Let V be a vector space over F, and let $S \subseteq V$.
Then FS is a subspace of V.

Proof. Use the subspace critereon (See book for proof).
Idea: The span FS is also the smallest vector space in V that contains S.

Given a (finite) set $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\} \subseteq V$, we might ask if some given v is in the span of S, i.e. if v is *generated* by S. In particular, we may ask if there exist $a_1, \dots, a_n \in F$ such that

$$v = a_1\mathbf{s}_1 + \cdots + a_n\mathbf{s}_n.$$

Example: Let $V = \mathbb{R}^3$ and $S = \{\mathbf{s}_1, \mathbf{s}_2\}$, where $\mathbf{s}_1 = (1, 2, 1)$ and $\mathbf{s}_2 = (5, 1, -1)$. To discover if $v = (-2, 5, 4)$ is in the span of S, we must solve

$$v = a_1\mathbf{s}_1 + a_2\mathbf{s}_2 \quad \text{for } a_1 \text{ and } a_2.$$

Namely, solve

$$(-2, 5, 4) = a_1(1, 2, 1) + a_2(5, 1, -1) = (a_1 + 5a_2, 2a_1 + a_2, a_1 - a_2).$$

This is equivalent to solving the system

$$\begin{cases} a_1 & + & 5a_2 & = & -2, \\ 2a_1 & + & a_2 & = & 5, \\ a_1 & + & (-1)a_2 & = & 4. \end{cases}$$

And reducing

$$\left(\begin{array}{cc|c} 1 & 5 & -2 \\ 2 & 1 & 5 \\ 1 & -1 & 4 \end{array}\right) \quad \text{yields} \quad \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right),$$

meaning $3\mathbf{s}_1 - \mathbf{s}_2 = v$, so that $v \in \mathbb{R}S$.

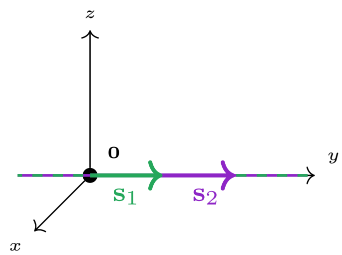
Redundancy in spanning sets

Consider

$$S = \{(1,0,0), (2,0,0)\} \subseteq \mathbb{R}^3.$$

The span of S in $\mathbb{R}^3$ is

$$\begin{aligned}\mathbb{R}S &= \{a(1,0,0) + b(2,0,0) \mid a, b \in \mathbb{R}\} \\ &= \{(a+2b, 0, 0) \mid a, b \in \mathbb{R}\} \\ &= \{(a+2b)(1,0,0) \mid a, b \in \mathbb{R}\} \\ &= \{c(1,0,0) \mid c \in \mathbb{R}\}.\end{aligned}$$



Geometrically: The vectors $(2,0,0)$ and $(1,0,0)$ are parallel, and so they both generate the same line.

Lemma. (Ch. 2, §II, Lemma 1.2) Let V be a v.s. over F, and let $S \subseteq V$. Then for any $v \in V$, we have

$$FS = F(S \cup \{v\}) \quad \text{if and only if} \quad v \in FS.$$

Sketch of proof. (See book for full proof)
Since $v \in F(S \cup \{v\})$, if $v \notin FS$, then it must be that $F(S \cup \{v\}) \neq FS$. Conversely, if $v \in FS$, then write $v = c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n$ and check that $FS \subseteq F(S \cup \{v\})$ and $F(S \cup \{v\}) \subseteq FS$ by direct computation. □

Linear (in)dependence

For the rest of today: Let V be a v.s. over a field F.

Let $S \subseteq V$. We say S is linearly dependent if there exist (distinct) $\mathbf{s}_1, \ldots, \mathbf{s}_n \in S$ and $c_1, \ldots, c_n \in F$ not all 0 (i.e. $c_i \neq 0$ for at least one i) such that

$$c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0}.$$

In other words, if there is more than the trivial way to build (or generate) $\mathbf{0}$ out of elements of S. If S is *not* linearly dependent, then we say S is linearly independent. Namely, S is linearly independent if

$$c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0} \quad \text{implies} \quad c_1 = \cdots = c_n = 0$$

for any $\mathbf{s}_1, \ldots, \mathbf{s}_n \in S$.

Ex. The set $S = \{(1,0,0), (2,0,0)\} \subseteq \mathbb{R}^3$ is linearly *dependent* because

$$2\begin{pmatrix}1\\0\\0\end{pmatrix} + (-1)\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Ex. The set $\{1, x, x^2\} \subseteq \mathbb{Q}[x]$ is linearly *independent* because

$$c_1 + c_2x + c_3x^2 = 0 \text{ (independent of } x\text{)} \quad \text{implies} \quad c_1 = c_2 = c_3 = 0.$$

Lemma. A set $S \subseteq V$ is linearly independent if and only if each element of FS has a *unique* expression as a linear combination of elements of S.

Note: This is closer to the book's definition, but they are equivalent.

To prove that a set S is *dependent*, you must find an example of a (non-trivial) linear combination building $\mathbf{0}$ out of distinct elements of S.

To prove that a set S is *independent*, you typically start by assuming that you have some linear combination of the form

$$c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0},$$

with $\mathbf{s}_1, \ldots, \mathbf{s}_n \in S$ distinct, and prove that $c_1, \ldots, c_n = 0$.

Either way, it's a reasonable start to assume $c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0}$ and try to solve for $c_1, \ldots, c_n$.

You try: For each of the following, decide whether the set if independent or dependent.

(a) $S = \left\{ \mathbf{s}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$

(b) $S = \{\mathbf{s}_1 = 1 + x,\ \mathbf{s}_2 = 1 + x + x^2,\ \mathbf{s}_3 = 1 - x\} \subseteq \mathbb{Q}[x]$

(c) $S = \{\mathbf{0}\}$

(d) $S = \{\mathbf{v}\}$ for any non-zero $\mathbf{v} \in V$

(e) $S = \{\mathbf{0}, \mathbf{v}\}$ for any non-zero $\mathbf{v} \in V$

(f) $S = \varnothing$

Prop. A set S is linearly dependent if and only if

$$\text{there exists } \mathbf{v} \in S \quad \text{such that} \quad \mathbf{v} \in F(S - \{\mathbf{v}\}).$$

"The set S is dependent exactly when it has superfluous elements."

Proof. If $\mathbf{v} \in S$ is in the span of $S - \{\mathbf{v}\}$, then

$$\mathbf{v} = c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n$$

for some $\mathbf{s}_1, \ldots, \mathbf{s}_n \in S - \{\mathbf{v}\}$. Hence

$$c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n + (-1)\mathbf{v} = \mathbf{0}$$

is a non-trivial linear combination of of elements of S. So S is linearly dependent.

Conversely, suppose

$$c_1\mathbf{s}_1 + \cdots + c_n\mathbf{s}_n = \mathbf{0} \qquad (*)$$

is a non-trivial linear combination. Then there is some ℓ for which $c_\ell \neq 0$. In particular,

$$-c_\ell\mathbf{s}_\ell = c_1\mathbf{s}_1 + \cdots + c_{\ell-1}\mathbf{s}_{\ell-1} + c_{\ell+1}\mathbf{s}_{\ell+1} + \cdots + c_n\mathbf{s}_n,$$

so that

$$\mathbf{s}_\ell = \sum_{\substack{i=1,\ldots,n \\ i\neq\ell}} \left(-\frac{c_i}{c_\ell}\right)\mathbf{s}_i \in F(S - \{\mathbf{s}_\ell\})$$

(i.e. we can solve for $\mathbf{s}_\ell$ in $(*)$). □

Prop. A set S is linearly dependent if and only if

$$\text{there exists } \mathbf{v} \in S \quad \text{such that} \quad \mathbf{v} \in F(S - \{\mathbf{v}\}).$$

"The set S is dependent exactly when it has superfluous elements."

Corollary. If $S \subseteq V$ is finite, then it contains a linearly independent subset.

Proof. Use induction on $|S|$, throwing elements out until you can't without changing the span.

Example.

$$S = \left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

Check in: $\mathbb{R}S \subseteq \mathbb{R}^3$. Why?

Look for linear dependence: solve

$$c_1 \begin{pmatrix}2\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + c_3 \begin{pmatrix}2\\2\\0\end{pmatrix} + c_4 \begin{pmatrix}0\\3\\1\end{pmatrix} + c_5 \begin{pmatrix}3\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Reduce

$$\left(\begin{array}{ccccc|c} 2 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 3/2 & 0 \\ 0 & 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

Solution set:

$$\left\{ \begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{pmatrix} = c_3 \begin{pmatrix}-1\\-2\\1\\0\\0\end{pmatrix} + c_5 \begin{pmatrix}-3/2\\3\\0\\-1\\1\end{pmatrix} \;\middle|\; c_3, c_5 \in \mathbb{R} \right\}$$

Choosing $c_3 = 1$ and $c_5 = 0$ (so that $c_1 = -1$, $c_2 = -2$, and $c_4 = 0$) tells us that

$$(-1) \begin{pmatrix}2\\0\\0\end{pmatrix} + (-2) \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}2\\2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \text{so that} \quad \begin{pmatrix}2\\2\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix} + 2 \begin{pmatrix}0\\1\\0\end{pmatrix},$$

and hence $\mathbb{R}S = \mathbb{R}(S - (2,2,0))$. Similarly, choosing $c_3 = 0$ and $c_5 = 1$ (so that $c_1 = -3/2$, $c_2 = 3$, and $c_4 = -1$) tells us that

$$\begin{pmatrix}3\\0\\1\end{pmatrix} = (3/2) \begin{pmatrix}2\\0\\0\end{pmatrix} + (-3) \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\3\\1\end{pmatrix} \in \mathbb{R}\left(S - \begin{pmatrix}2\\2\\0\end{pmatrix} - \begin{pmatrix}0\\3\\1\end{pmatrix} \right).$$

So

$$\mathbb{R}\left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix} \right\} = \mathbb{R}\left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix} \right\}$$

Example.

$$S = \left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

Reduce

$$\left(\begin{array}{ccccc|c} 2 & 0 & 2 & 0 & 3 & \not{0}\ b_1 \\ 0 & 1 & 2 & 3 & 0 & \not{0}\ b_2 \\ 0 & 0 & 0 & 1 & 1 & \not{0}\ b_3 \end{array}\right) \mapsto \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 3/2 & \not{0}\ b_1/2 \\ 0 & 1 & 2 & 0 & -3 & \not{0}\ b_2 - 3b_3 \\ 0 & 0 & 0 & 1 & 1 & \not{0}\ b_3 \end{array}\right)$$

Discard $(2,2,0)$ and $(3,0,1)$ to get $\mathbb{R}S = \mathbb{R}S'$, where $S' = \left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix} \right\}$.

Big things to notice:

1. We ended up
   - discarding vectors corresponding to free variables, and
   - keeping the vectors corresponding to pivot terms.

   This will always work. (Why?)
   So in retrospect, we could have spotted a linearly independent subset from the point that we reached row echelon form (no need to reduce)!
2. Columns don't interact with each other in row operations, so I've already reduced the "sub-array" corresponding to the subset S'. Outcome:
   - We're done! Discarding the "free columns" leaves a lin. indep. set.

   Observe, in *this* example, changing the 0's in the right-hand column to other constants, I reduce to a solvable system! $\boxed{\mathbb{R}S' = \mathbb{R}^3}$

Bases

Still: Let V be a vector space over F.

A spanning set for V is a subset $S \subseteq V$ such that $FS = V$.

Think: S is "enough" to build/generate V.

A basis for V is a linearly independent spanning set.

Think: S is *minimal* in being "enough" to build/generate V.

Example: We just showed that $\left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Example: Another basis of $\mathbb{R}^3$ is the natural or canonical basis

$$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad \text{where} \quad \mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Consider for next time:

1. We have lots of examples of vector spaces now. Can you come up with a basis for each of them? For example, can you find a basis for...
   - $V = \mathcal{P}_n(\mathbb{R}) = \{f \in \mathbb{R}[x] \mid \deg(f) \leqslant n\}$ over $F = \mathbb{R}$;
   - $V = M_{2,2}(\mathbb{R}) = \{$ 2×2 matrices w/ coefs in $\mathbb{R}$ $\}$, over $F = \mathbb{R}$;
   - $V = 0 = \{\mathbf{0}\}$, the trivial vector space over a field F;
2. Many vector spaces have more than one basis. Under what circumstances will a basis of V be unique?
3. Does every vector space even *have* a basis? (How could you prove or disprove?)

Epilog: Some tips for translating between lecture and the book.

- The notation for span in the book is $[S]$.
  - Pro: this is one common mathematical notation for the "*closure*" of a set.
  - Cons:
    - We'll need [ ] elsewhere later;
    - $[S]$ is not as ubiquitous in the literature;
    - $[S]$ doesn't specify the field.
- The book is using $\vec{v}$ for vectors; LaTeX: `\vec{v}`.
  We're using $\mathbf{v}$; LaTeX: `\mathbf{v}` or `\vv` using my shortcuts.
  Either is fine for the homework. The arrow notation is great for hand-written math (boldface is harder to write); I avoid it in LaTeX because it can get aesthetically busy and can align strangely on some symbols. *But*, the book is about to start using greek letters for basis vectors...
- The book defines a basis as an *ordered* set (that spans and is independent)—i.e. what we're calling an *ordered* basis. This isn't standard, so we'll differentiate between the two. It also notationally distinguishes between a set:
  $$B = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\}; \qquad \text{(set)}$$
  and an *ordered* set (a list):
  $$B = \langle\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\rangle. \qquad \text{(ordered set)}$$
  There is no standard convention for what to use for notation in the second case.