Lecture 2

Warmup:

Consider the function $f(x) = ax^2 + bx + c$. Suppose I know that $f(x)$ passes through the points

$$p_1 = (1, 0), \quad p_2 = (-1, 6), \quad \text{and} \quad p_3 = (2, 3).$$

Plugging in the point $p_1 = (1, 0)$ gives me the equation

$$a(1)^2 + b(1) + c = 0, \quad \text{i.e.} \quad a + b + c = 0.$$

Note that this is a linear equation in the *variables* a, b, and c (even through $f(x)$ thinks of x as its variable and a, b, and c as constants).

1. Plug in the points p_2 and p_3 to get two more linear equations.
2. Write the augmented matrix associated to the three linear equations we've found.
3. Put the associated augmented matrix into reduced echelon form (where every pivot is a 1 and every other entry in the same column is a 0).
4. What quadratic function passes through the points p_1, p_2, and p_3?
5. How many points would you need to know to in order to uniquely identify the coefficients of a polynomial of degree n (or less), i.e. of the form $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$?

(Column) vectors

We call an $n \times 1$ matrix a *(column) vector*:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Two kinds of operations involving vectors that we'll use a *lot* are addition and scaling, both performed entry-by-entry:

addition: $$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$ (vector dimensions must match)

scaling: $$\beta \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \beta a_1 \\ \beta a_2 \\ \vdots \\ \beta a_n \end{pmatrix}$$

(where $a_i, b_i, \beta \in \mathbb{R}$ are all real numbers... for now).

Back to our warmup

Note that if I *only* knew that $f(x) = ax^2 + bx + c$ passed through

$$p_1 = (1, 0) \quad \text{and} \quad p_2 = (-1, 6),$$

then I would have only had two equations:

$$\begin{cases} a + b + c = 0, \\ a - b + c = 6. \end{cases}$$

The associated augmented matrix and its reduced row echelon form are

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 6 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3 \end{array}\right),$$

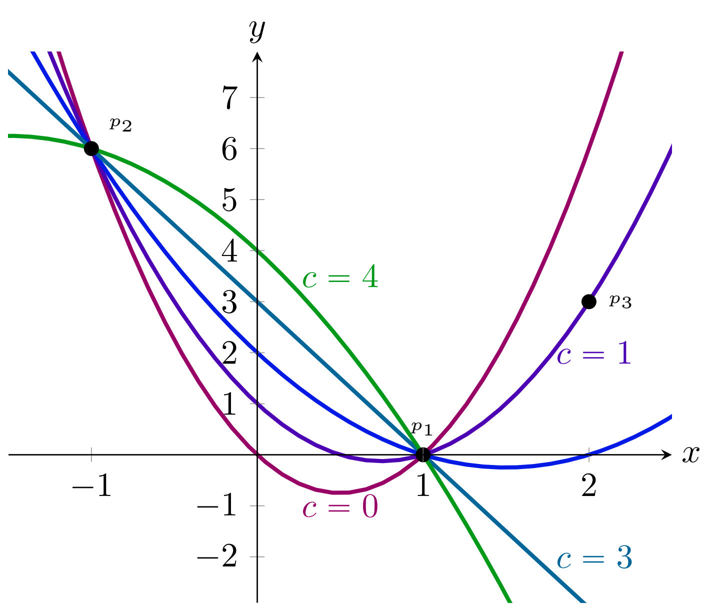
respectively. Converting back into a system of equations, we have

$$\begin{cases} a + c = 3, \\ \quad\ b = -3, \end{cases} \quad \text{i.e.} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 + c(-1) \\ -3 + c(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{cases} a + c = 3, \\ \quad\ b = -3, \\ \quad\ c = c, \end{cases} \quad \text{i.e.} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 + c(-1) \\ -3 + c(0) \\ 0 + c(1) \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So there is an infinite family of functions of the form $f(x) = ax^2 + bx + c$ that pass through $p_1 = (1, 0)$ and $p_2 = (-1, 6)$, depending on the parameter/free variable c:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{so that} \quad f(x) = (3 - c)x^2 - 3x + c.$$



Reduced (row) echelon form:

$$\left(\begin{array}{ccccccccc|c}
\triangledown & \star & \triangledown & \triangledown & \star & \star & \triangledown & \star & \star & \\
1 & 2 & 0 & 0 & 4 & 17 & 0 & -6 & 0 & 5 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -7 & 3 & -3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$\triangledown$: leading variable

$\star$: free variable

Corresponding system of equations:

$$\begin{cases} x_1 + 2x_2 + 4x_5 + 17x_6 - 6x_8 = 5 \\ x_3 - 7x_8 + 3x_9 = -3 \\ x_4 - 3x_8 + x_9 = -1 \\ x_7 + \frac{1}{2}x_9 = \frac{1}{2} \end{cases}$$

Solving for leading variables:

$$\begin{aligned} x_1 &= 5 - 2x_2 - 4x_5 - 17x_6 + 6x_8 \\ x_3 &= -3 + 7x_8 - 3x_9 \\ x_4 &= -1 + 3x_8 - x_9 \\ x_7 &= \tfrac{1}{2} - \tfrac{1}{2}x_9 \end{aligned}$$

(careful with signs!)

The solution set (in set notation):

$$\left\{ \left(\overbrace{5 - 2x_2 - 4x_5 - 17x_6 + 6x_8}^{x_1},\ x_2,\ \overbrace{-3 + 7x_8 - 3x_9}^{x_3},\ \underbrace{-1 + 3x_8 - x_9}_{x_4},\ x_5,\ x_6,\ \underbrace{\tfrac{1}{2} - \tfrac{1}{2}x_9}_{x_7}, x_8, x_9 \right) \;\middle|\; x_2, x_5, x_6, x_8, x_9 \in \mathbb{R} \right\}$$

The solution set (in column vector form):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -3 \\ -1 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -17 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_8 \begin{pmatrix} 6 \\ 0 \\ 7 \\ 3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_9 \begin{pmatrix} 0 \\ 0 \\ -3 \\ -1 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix},$$

$x_2, x_5, x_6, x_8, x_9 \in \mathbb{R}$.

Example

Suppose your linear system has reduced echelon form

$$A = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

1. Describe the solution sets both in set notation

$$\{\,(\_\_, \_\_, \_\_, \_\_) \,|\, \_\_\_\_\_\_ \in \mathbb{R}\}$$

   and in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \cdots$$

2. Give a few examples of points in your solution set, and test them. [Namely, pick a couple of examples of values for all your free variables (e.g. $x_{??} = 2$). Then check: do they satisfy the system of equations associated to the matrix A?]

Remark: In the solution set

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$$

we can get one very special solution, called the particular solution, by setting $x_2 = x_4 = 0$:

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 6 \\ 0 \end{pmatrix}} + 0 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix}.$$

In general, the general solution is where we treat all the free variables as parameters/variables, and the particular solution is where we set all the free variables to 0.

Note: A system doesn't have a unique particular solution *until* we choose the order of our variables.

Challenge example: Go back to the augmented matrix in our last example, but now put the columns in the order x_4, x_3, x_2, x_1. You should see that your matrix isn't in reduced echelon form anymore! Put it in reduced form (use computational software?), and re-write your solution set. What's the "particular solution" now? Check that your particular solution is still a specific example of the general solution we found above. (What values to the free variables take on to get this solution?)

Geometry in $\mathbb{R}^n$

Recall that we write $\mathbb{R}$ to mean the set of real numbers. Then

$$\mathbb{R}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}.$$

Note: Depending on context, we may list points as n-tuples, or we might write them as column vectors

$$(a_1, \ldots, a_n) \quad \longleftrightarrow \quad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

(There are Good Reasons for this correspondence... we'll get there.)

We also *think* of elements $\mathbf{v} \in \mathbb{R}^n$ as two different kinds of physical objects: as points and as vectors (an arrow with a magnitude and direction corresponding to starting at the origin and pointing to the point $\mathbf{v}$).

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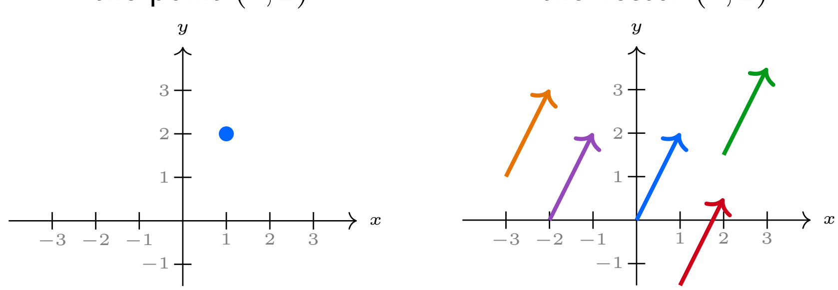
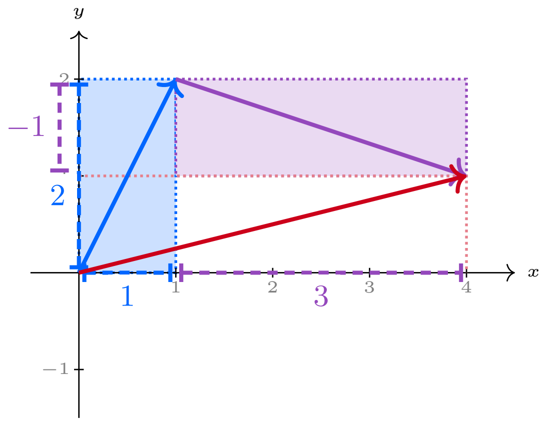
the point $(1, 2)$ | the vector $(1, 2)$



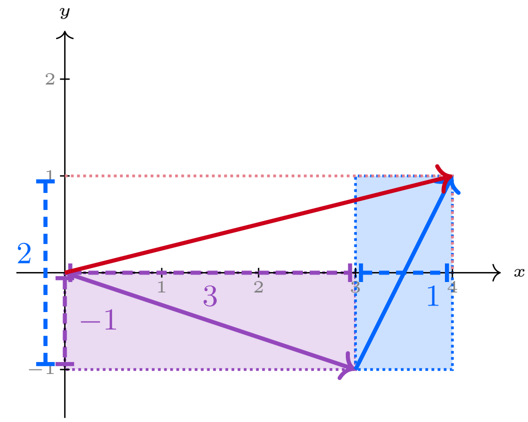
(vectors don't depend on placement)

Geometry of vector addition and scaling

$$\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$$



$$\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$$

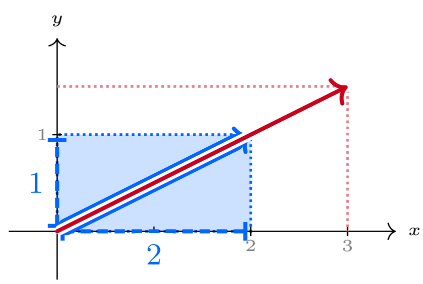


Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. In general...

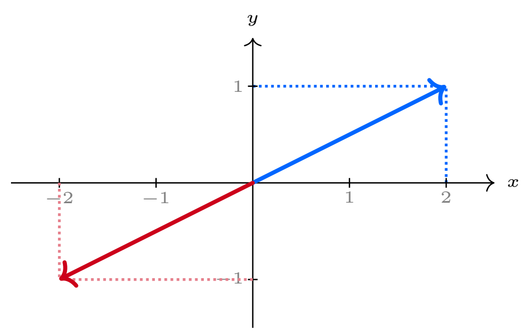
- Geometrically, the vector $\mathbf{u}+\mathbf{v}$ looks like the result of traveling along $\mathbf{u}$, followed by traveling along $\mathbf{v}$.
- Addition is commutative, even when we think geometrically.

Geometry of vector addition and scaling

$$1.5\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}3\\1.5\end{pmatrix}$$



$$(-1)\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\end{pmatrix}$$



Let $\mathbf{u}$ be a vector. In general, geometrically speaking...

- If $c>0$ is a real number, then $c\mathbf{u}$ is the result of stretching (or squishing) the magnitude of $\mathbf{u}$ by c.
- The vector $-\mathbf{u}$ is the result of traveling backwards along $\mathbf{u}$.
- For $c<0$, we have $c=-|c|$. So geometrically, $c\mathbf{u}$ is traveling backwards along $|c|\mathbf{u}$.

You try:

1. Convince yourself both algebraically and geometrically that $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$, where $\mathbf{0}$ is the vector of all 0's. [Algebraically means start with $\mathbf{u} = (u_1, \ldots, u_n)$ and compute $(-1)\mathbf{u}$ and then $\mathbf{u} + (-1)\mathbf{u}$ directly.]

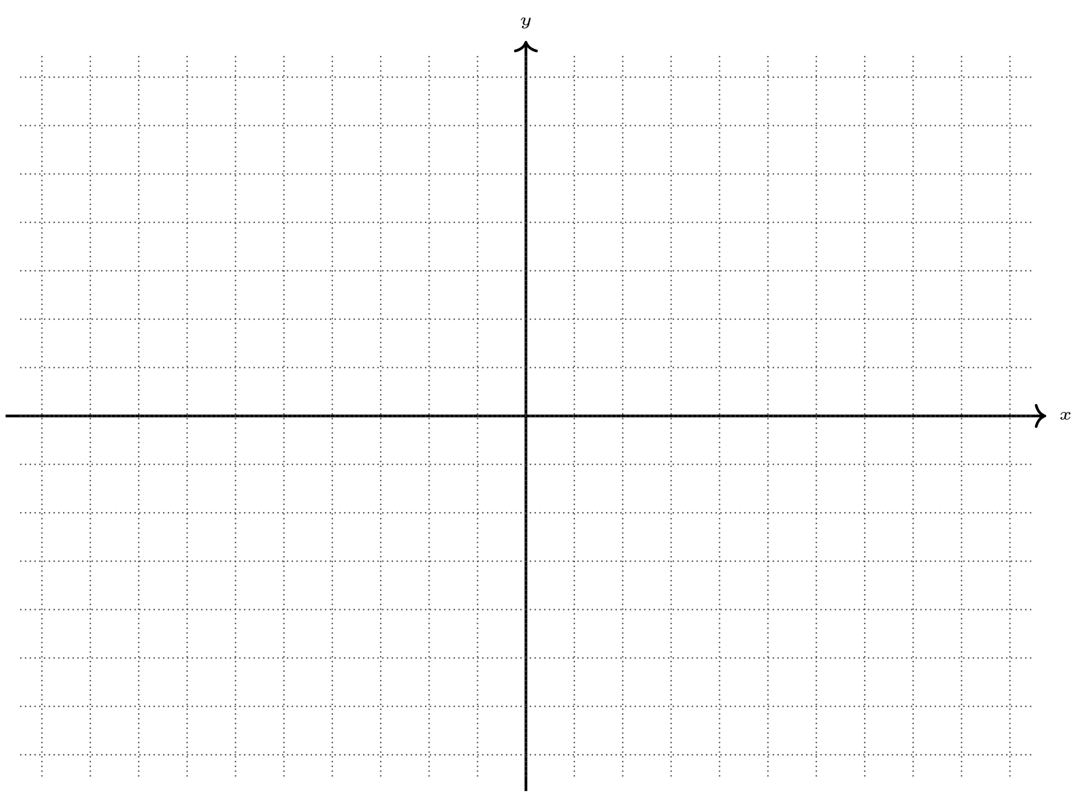
2. Using order-of-operations, $\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u})$ means "add $\mathbf{v}$ to backwards-of-$\mathbf{u}$". Convince yourself that $\mathbf{v} - \mathbf{u}$ is the vector pointing from the tip of $\mathbf{u}$ to the tip of $\mathbf{v}$ (draw pictures).

3. Consider the vectors $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (3, -1)$. Compute the points

$$\mathbf{u} + t\mathbf{v} \quad \text{for } t = -2, -1, 0, 1, \text{ and } 2,$$

and then plot all five points on the same axis. Hypothesize about the shape of the set

$$\{(2, 1) + t(3, -1) \mid t \in \mathbb{R}\}.$$



Lines in $\mathbb{R}^n$

In 2 dimensions, we have plenty of ways to express lines (e.g. $y = mx + b$). But in more dimensions, we need vectors!

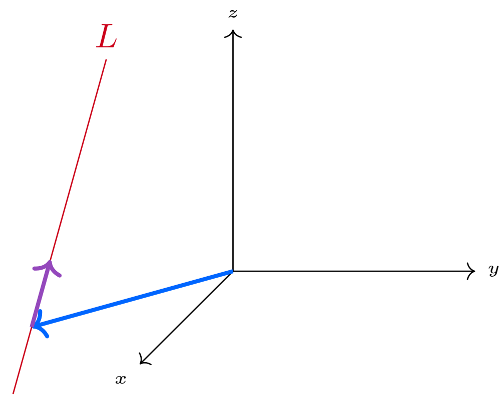
Prop. Let $\mathbf{p}, \mathbf{d} \in \mathbb{R}^n$ with $\mathbf{d} \neq \mathbf{0}$ (the vector filled with 0's). Then the set

$$L = \{\mathbf{p} + t\mathbf{d} \mid t \in \mathbb{R}\}$$

is a line traveling through the point $\mathbf{p}$, in the direction of $\mathbf{d}$.

Example in 3 dimensions:

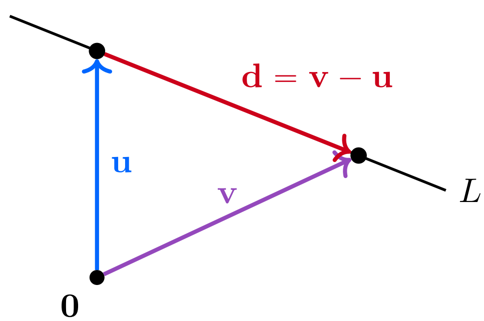
$$L = \left\{ \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \; t \in \mathbb{R} \right\}$$



Another way to think about this: If a system of equations has exactly one free variable, then its solution set is a line! ("One dimension of freedom")

Constructing a parametric equation for a line from points

Working backwards, since $\mathbf{v} - \mathbf{u}$ is a vector pointing from the point $\mathbf{u}$ to the point $\mathbf{v}$, it is a direction vector for the line passing though $\mathbf{u}$ and $\mathbf{v}$.



$$L = \{\mathbf{u} + t\mathbf{d} \mid t \in \mathbb{R}\}$$

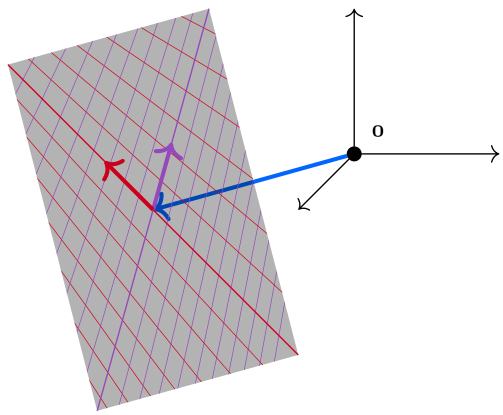
Ex: Find an equation for the line that passes through the points $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 2, 3)$. [Compute $\mathbf{d} = \mathbf{v} - \mathbf{u}$ and use $\mathbf{p} = \mathbf{u}$.]

Planes

Two “dimensions of freedom”/parameters:

$$H = \{\mathbf{p} + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$$

Useful metaphor: Think of $\mathbf{u}$ and $\mathbf{v}$ as directions of city streets—North/South and East/West. The parameter s says how far you travel along $\mathbf{u}$ and the parameter t says how far you travel along $\mathbf{v}$.



Further reading: In special circumstances we can convert

2 variables:	parametric lines	$\longleftrightarrow$	standard equations for lines
3 variables:	parametric planes	$\longleftrightarrow$	standard equations for planes