Welcome to

Math 201: Linear Algebra

w/ prof. Zajj Daugherty

Logistics

Where to find class stuff:

- **Moodle** has course files, notes, outline, syllabus, etc.
- **Gradescope** is where you'll turn in homework and exams.
- **Slack** is where we'll all correspond throughout the week. (Messaging me there is preferred to emails.)

To do list:

- Moodle
- Read the syllabus (be ready to discuss next time)
- Get the book, and check out the book's website(s)
- Get set up with LaTeX, Slack, and Gradescope (click on HW 0)
- Fill out "Getting to know you" survey
- Do HW 0

Suppose you want to solve the system of equations

$$2x + 3y = 13$$
$$5x - 5y = -5$$

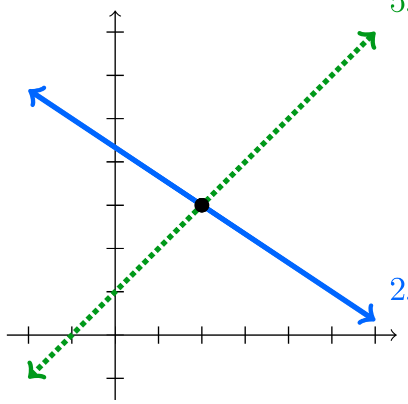
"solve": find values for x and y that satisfy both equations simultaneously

How?*

*or how do we know if we can solve this system at all?

*and are there lots of solutions? just one?

Graphing/Guessing



A "solution" to the system is a point that sits on both lines at the same time, i.e. a point of intersection.

Solve for a variable and plug in...

First,

$$5x - 5y = -5 \quad \text{is equivalent to} \quad y = x + 1$$

Plug $y = x + 1$ into $2x + 3y = 13$ gives

$$13 = 2x + 3(x + 1) = 5x + 3, \qquad \text{so that } 5x = 10.$$

Hence $x = 2$ and $y = 2 + 1 = 3$.

Bigger systems

Suppose you want to solve the system of equations

$$\begin{aligned}
32x_1 + 9x_2 - 2/3x_3 + 100.4x_4 - 16/7x_6 + 83x_7 - 23,501x_9 &= \pi \\
-53\sqrt{82}x_2 + x_3 - 77.23x_5 + x_8 &= 5 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 &= 55 - \sqrt{5} \\
-55x_1 + .0001x_2 + .139x_3 - 17x_4 + x_6 &= 0 \\
x_1 - x_2 + x_8 - x_9 &= 2 \\
x_1 - 87x_5 + x_6 + -x_9 &= 500,000 \\
x_3 + 2x_5 - \sqrt{57}x_8 + x_9 &= 0 \\
x_4 + x_5 + x_6 + x_7 - x_8 - x_9 &= 3 \\
538x_2 - 374.25x_7 - \sqrt{3}x_8 &= 15.2
\end{aligned}$$

How?*

*or how do we know if we can solve this system at all?
*and are there lots of solutions? just one?

Gaussian elimination (Thm. 1.5)

Make operations between the equations *as a system*, instead of manipulating the equations individually.

Three types of row operations:

> Tip: This book refers to the equations by $\rho_1, \rho_2, \ldots,$ counting top-to-bottom, for short. Think "rho one" = "row one".

1. multiply an equation by a nonzero scalar, "rescaling"

$$\begin{cases} 2x + 3y = 13 \\ 5x - 5y = -5 \end{cases} \xrightarrow{\text{multiply } \rho_2 \text{ by } 1/5} \begin{cases} 2x + 3y = 13 \\ x - y = -1 \end{cases}$$

2. swap two equations, or "swapping"

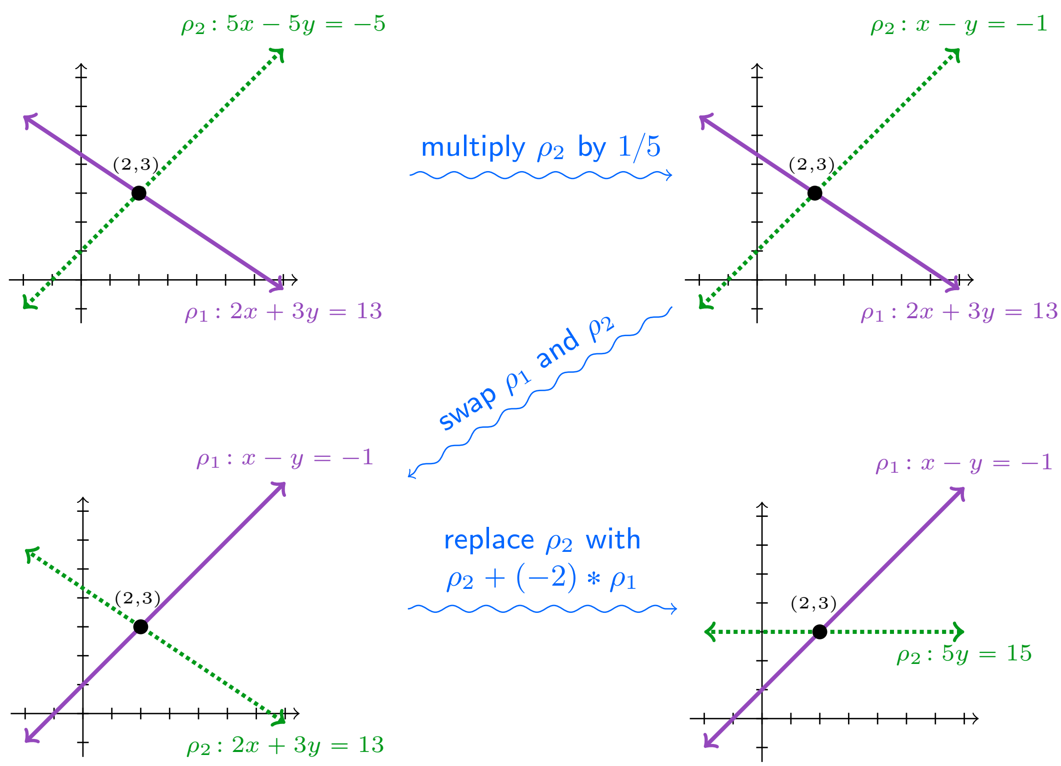
$$\begin{cases} 2x + 3y = 13 \\ x - y = -1 \end{cases} \xrightarrow{\text{swap } \rho_1 \text{ and } \rho_2} \begin{cases} x - y = -1 \\ 2x + 3y = 13 \end{cases}$$

3. replace an equation with it *plus* a multiple of another. "row combination"

$$\begin{cases} x - y = -1 \\ 2x + 3y = 13 \end{cases} \xrightarrow{\text{replace } \rho_2 \text{ with } \rho_2 + (-2) * \rho_1} \begin{cases} x - y = -1 \\ 5y = 15 \end{cases}$$

Hey! We eliminated x from ρ_2!

What do each of these moves do to our graph?



Finishing up our small example:

$$\begin{cases} 2x + 3y = 13 \\ 5x - 5y = -5 \end{cases} \xrightarrow{\rho_2 \mapsto (1/5)\rho_2} \begin{cases} 2x + 3y = 13 \\ x - y = -1 \end{cases}$$

$$\xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{cases} x - y = -1 \\ 2x + 3y = 13 \end{cases}$$

$$\xrightarrow{\rho_2 \mapsto \rho_2 + (-2)\rho_1} \begin{cases} x - y = -1 \\ 5y = 15 \end{cases}$$

$$\xrightarrow{\rho_2 \mapsto (1/5)\rho_2} \begin{cases} x - y = -1 \\ y = 3 \end{cases}$$

$$\xrightarrow{\rho_1 \mapsto \rho_1 + \rho_2} \begin{cases} x = 2 \\ y = 3 \end{cases}$$

This is our solution!

Row operations:
1. rescaling,
2. swapping,
3. row combination.

Key:
- ρ_i means "row i",
- $\mapsto$ means "becomes",
- $\leftrightarrow$ means "swaps with".

You try

Think/work for 5-10 minutes:

1. Consider systems with two variables and two "linear" equations

$$\begin{cases} c_1x + c_2y = c_3, \\ d_1x + d_2y = d_3 \end{cases}$$

(where c_1, c_2, c_3, d_1, d_2, and d_3 are all constants, with c_1 and c_2 not both equal to 0, and d_1 and d_2 not both equal to 0).

   Think graphically/geometrically: a solution to this system is a point that is on both lines at once. Is there necessarily a solution? Is the solution necessarily unique? What are the cases, and how would you describe them in terms of the graphs of the two lines? Draw!

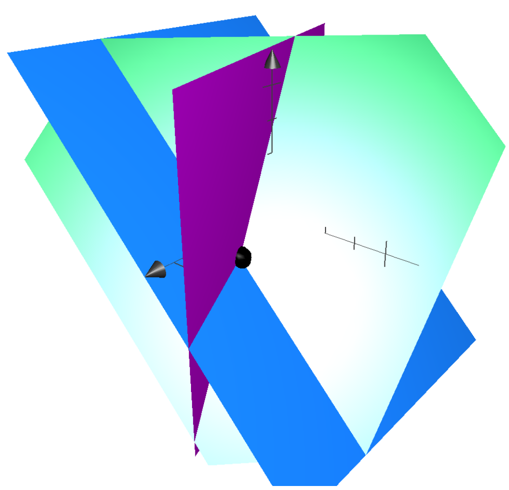
2. Use Gaussian elimination/row operations to solve the system

$$\begin{cases} x & & & - & z & = & 0 \\ 3x & + & y & & & = & 1 \\ -x & + & y & + & z & = & 4 \end{cases}$$

   (there is a unique solution). Be sure to practice writing *one* operation at a time, labeling your steps.

3. If you got an answer to #2 (something of the form $x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$, and $z = \underline{\ ?\ }$), plug your answer in to each of the three equations to *check* that your solution is correct.

$$\begin{cases} x & & & - & z & = & 0 \\ 3x & + & y & & & = & 1 \\ -x & + & y & + & z & = & 4 \end{cases}$$



Linear systems

A linear equation is one of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $x_1, \ldots, x_n$ are (unknown) variables and $a_1, \ldots, a_n, b$ are (known) coefficients/constants.

Yes, linear:	No, not linear:
$x_1 = 1$	$x_1^2 = 1$
$x_1 + x_3 = 15$	$x_1x_2 - x_3 = 15$
$x_2 - \cos(67)\ x_5 + \pi\sqrt{-2}\ x_{153} = 0$	$\sin(x_1) - \sqrt{x_2} = 0$

A system of linear equations, or a linear system is just a collection of linear equations (technically, all in the same set of variables). A solution to a linear system in variables $x_1, \ldots, x_n$ is a collection of values $x_1 = c_1, \ldots, x_n = c_n$ that satisfies all of the equations in the system simultaneously.

Theorem (Ch 1. Thm. 1.5, Gauss's Method)

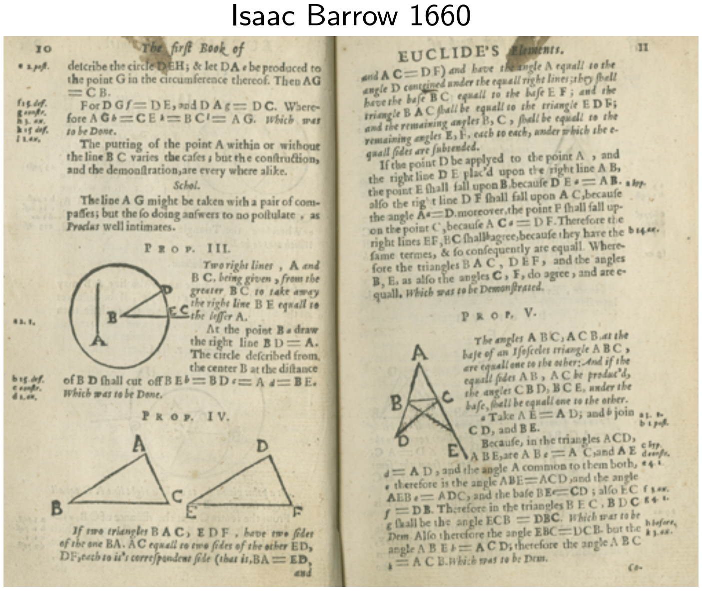
If a linear system is changed to another by one of the three row operations (rescaling, swapping, or row combination), then the two systems have exactly the same set of solutions.

Strategy: Use row operations to simplify your system until you find its solution(s) or arrive at a contradiction.

Math = repackaging

"Euclid's elements" $\sim$ 300 BCE

Isaac Barrow 1660



10 *The firſt Book of*

describe the circle DEH; & let DA *e* be produced to the point G in the circumference thereof. Then AG = CB.

For DG *f* = DE, and DA *g* = DC. Wherefore AG *h* = CE *k* = BC *l* = AG. *Which was to be Done.*

The putting of the point A within or without the line BC varies the caſes; but the conſtruction, and the demonſtration, are every where alike.

Schol.

The line AG might be taken with a pair of compaſſes; but the ſo doing anſwers to no poſtulate, as *Proclus* well intimates.

PROP. III.

Two right lines, A and BC, being given, from the greater BC to take away the right line BE equall to the leſſer A.

At the point B *a* draw the right line BD = A. The circle deſcribed from the center B at the diſtance of BD ſhall cut off BE *b* = BD *c* = A *d* = BE. *Which was to be Done.*

PROP. IV.

If two triangles BAC, EDF, have two ſides of the one BA, AC equall to two ſides of the other ED, DF, each to it's correſpondent ſide (that is, BA = ED, and

EUCLIDE'S *Elements.* 11

and AC = DF) and have the angle A equall to the angle D contained under the equall right lines; they ſhall have the baſe BC equall to the baſe EF; and the triangle BAC ſhall be equall to the triangle EDF; and the remaining angles B, C, ſhall be equall to the remaining angles E, F, each to each, under which the equall ſides are ſubtended.

If the point D be applyed to the point A, and the right line DE plac'd upon the right line AB, the point E ſhall fall upon B, becauſe DE *a* = AB. alſo the right line DF ſhall fall upon AC, becauſe the angle A *a* = D. moreover, the point F ſhall fall upon the point C, becauſe AC *a* = DF. Therefore the right lines EF, BC ſhall *b* agree, becauſe they have the ſame termes, & ſo conſequently are equall. Wherefore the triangles BAC, DEF, and the angles B, E, as alſo the angles C, F, do agree, and are equall. *Which was to be Demonſtrated.*

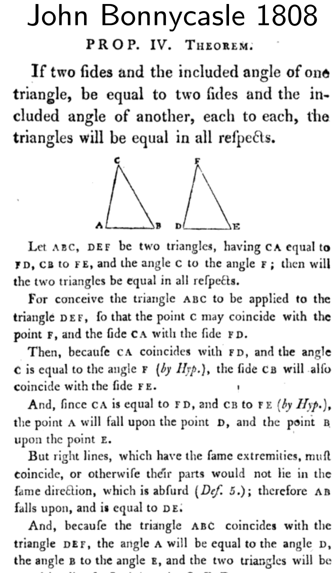
PROP. V.

The angles ABC, ACB, at the baſe of an Iſoſceles triangle ABC, are equall one to the other: And if the equall ſides AB, AC be produc'd, the angles CBD, BCE, under the baſe, ſhall be equall one to the other.

a Take AE = AD; and *b* join CD, and BE.

Becauſe, in the triangles ACD, ABE, are AB *c* = AC, and AE *d* = AD, and the angle A common to them both, *e* therefore is the angle ABE = ACD, and the angle AEB *e* = ADC, and the baſe BE = CD; alſo EC *f* = DB. Therefore in the triangles BEC, BDC *g* ſhall be the angle ECB = DBC. *Which was to be Dem.* Alſo therefore the angle EBC = DCB. but the angle ABE *h* = ACD; therefore the angle ABC *k* = ACB. *Which was to be Dem.*

John Bonnycasle 1808



PROP. IV. THEOREM.

If two ſides and the included angle of one triangle, be equal to two ſides and the included angle of another, each to each, the triangles will be equal in all reſpects.

Let ABC, DEF be two triangles, having CA equal to FD, CB to FE, and the angle C to the angle F; then will the two triangles be equal in all reſpects.

For conceive the triangle ABC to be applied to the triangle DEF, ſo that the point C may coincide with the point F, and the ſide CA with the ſide FD.

Then, becauſe CA coincides with FD, and the angle C is equal to the angle F (*by Hyp.*), the ſide CB will alſo coincide with the ſide FE.

And, ſince CA is equal to FD, and CB to FE (*by Hyp.*), the point A will fall upon the point D, and the point B upon the point E.

But right lines, which have the ſame extremities, muſt coincide, or otherwiſe their parts would not lie in the ſame direction, which is abſurd (*Def.* 5.); therefore AB falls upon, and is equal to DE.

And, becauſe the triangle ABC coincides with the triangle DEF, the angle A will be equal to the angle D, the angle B to the angle E, and the two triangles will be equal in all reſpects (*Ax.* 9.) Q. E. D.

Prop. Let $\triangle ABC$ and $\triangle DEF$ be triangles. If $|AB| = |DE|$, $|AC| = |DF|$, and $\measuredangle CAB = \measuredangle FDE$, then $\triangle ABC = \triangle DEF$.

Proof. Since $|AB| = |DE|$, we can overlay the vertices A and B with D and E, respectively. But now, since $\measuredangle CAB = \measuredangle FDE$ and $|AC| = |DF|$, we know vertices C and F match up as well. So $\triangle ABC = \triangle DEF$. $\square$

Representing linear systems as matrices

The matrix associated to a linear system

$$\begin{aligned} a_{1,1}x_1 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad\qquad\quad \vdots \;\; &\;\; \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_m &= b_m \end{aligned}$$

is the (augmented) $m \times (n|1)$ array

$$\left(\begin{array}{ccc|c} a_{1,1} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & \cdots & a_{2,n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m,1} & \cdots & a_{m,n} & b_m \end{array}\right)$$

Warning: There are Reasonable Reasons that the matrix coordinates are *not* labeled with the same conventions as Cartesian coordinates (we'll get there). But they're not!

Row operations (same as before)

1. Rescaling

$$\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ a_{\ell,1} & a_{\ell,2} & \cdots & a_{\ell,n} & b_\ell \\ * & * & \cdots & * & * \end{array}\right) \xmapsto{\rho_\ell \mapsto c\rho_\ell} \left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ ca_{\ell,1} & ca_{\ell,2} & \cdots & ca_{\ell,n} & cb_\ell \\ * & * & \cdots & * & * \end{array}\right)$$

2. Swapping

$$\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} & b_k \\ * & * & \cdots & * & * \\ a_{\ell,1} & a_{\ell,2} & \cdots & a_{\ell,n} & b_\ell \\ * & * & \cdots & * & * \end{array}\right) \xmapsto{\rho_k \leftrightarrow \rho_\ell} \left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ a_{\ell,1} & a_{\ell,2} & \cdots & a_{\ell,n} & b_\ell \\ * & * & \cdots & * & * \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} & b_k \\ * & * & \cdots & * & * \end{array}\right)$$

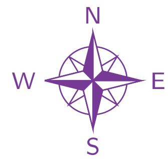
3. Row combination

$$\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} & b_k \\ * & * & \cdots & * & * \\ a_{\ell,1} & a_{\ell,2} & \cdots & a_{\ell,n} & b_\ell \\ * & * & \cdots & * & * \end{array}\right)$$

$$\xmapsto{\rho_\ell \mapsto \rho_\ell + c\rho_k} \left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} & b_k \\ * & * & \cdots & * & * \\ a_{\ell,1} + ca_{k,1} & a_{\ell,2} + ca_{k,2} & \cdots & a_{\ell,n} + ca_{k,n} & b_\ell + cb_k \\ * & * & \cdots & * & * \end{array}\right)$$

Echelon form—reduced and otherwise

A matrix is in echelon form if the leading term (the left-most non-zero term) in each row has only zeros southwest of it (0's north and east are ok too). By convention, rows of all 0's must be at the bottom.



Example:

$$\begin{pmatrix} 1 & 5 & 0 & -2 & 3 & 17 & 4 & 0 & 0 & 9 \\ 0 & 0 & 6 & -10 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 1 & 1 \\ 0 & 0 & 0 & 14 & 0 & 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Echelon form—reduced and otherwise

A matrix is in echelon form if the leading term (the left-most non-zero term) in each row has only zeros southwest of it (0's north and east are ok too). By convention, rows of all 0's must be at the bottom.

Example:

$$\begin{pmatrix} \boxed{1} & 5 & 0 & -2 & 3 & 17 & 4 & 0 & 0 & 9 \\ 0 & 0 & \boxed{6} & -10 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & \boxed{14} & 0 & 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-5} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

i. Move row 3 to row 5.*

ii. Swap row 3 and 4.

In echelon form, the leading terms are also called pivot entries or pivots.

Echelon form—reduced and otherwise

A matrix is in echelon form if the leading term (the left-most non-zero term) in each row has only zeros southwest of it (0's north and east are ok too). By convention, rows of all 0's must be at the bottom.

A matrix is in reduced echelon form if

1. it is in echelon form;
2. the leading terms/pivots are all 1's (rows of all 0's are ok); **and**
3. each leading term is the only non-zero term in its column.

For example: Each of these is in echelon form.

Exactly one is *reduced*—which one?

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Algorithm:

- Moving left-to-right, move a row with a left-most leading term up as far as possible.
- Normalize (rescale) so that the leading term is 1.
- Use that row to "clear" non-zero terms in the same column.
- Repeat using the next left-most leading term.

Example: Put the following matrix into reduced echelon form.

$$\begin{pmatrix} 1 & 2 & 2 & -1 & 7 \\ 0 & 0 & 2 & -1 & 8 \\ 4 & 8 & 2 & -5 & 12 \end{pmatrix}$$

You try:

1. Consider the systems

$$\text{(I)} \begin{cases} x + y + z = 5 \\ x - y = 0 \\ y + 2z = 7 \end{cases} \qquad \text{(II)} \begin{cases} x + z = 4 \\ x - y + 2z = 5 \\ 4x - y + 5z = 17 \end{cases}$$

   For each of the systems (a) convert it into its corresponding (augmented) matrix; (b) put the matrix into reduced echelon form; and (c) convert back into a system of linear equations.

   What can you say about the system's solutions? Does one exist? Is there more than one solution?

2. (Time-permitting) What "shortcuts" can we take with row operations? What shortcuts get us into trouble?

   For example, I can perform the shortcut of moving row 1 into the third row, and moving rows 2 and 3 up by one by doing $\rho_1 \leftrightarrow \rho_2$ followed by $\rho_2 \leftrightarrow \rho_3$.

   I *can't* always do multiple row combinations though. For example, performing $\rho_1 \mapsto \rho_1 + \rho_2$ and $\rho_2 \mapsto \rho_1 + \rho_2$ won't preserve the solution sets of the system

$$\begin{cases} 2x + 3y = 13 \\ x - y = -5 \end{cases} \qquad \text{(try it!)}$$

   Brainstorm how else operations can be safely combined, and what kinds of combinations can get you into trouble. Are any of our basic moves redundant? Try some examples!

The system

$$\left\{ \begin{aligned} x + z &= 4, \\ x - y + 2z &= 5, \\ 4x - y + 5z &= 17, \end{aligned} \right. \qquad \text{has reduced aug. matrix} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$\text{which means} \quad \left\{ \begin{aligned} x + z &= 4, \\ y - z &= -1, \\ 0 &= 0. \end{aligned} \right.$$

This system has infinitely many solutions! We can write

$$\left\{ \begin{aligned} x &= 4 - z, \\ y &= -1 + z, \end{aligned} \right. \qquad \text{or} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + z \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We call x and y leading variables and z a free variable.

In general, for a system that's in reduced echelon form, the variables corresponding to pivots are the leading variables, and everything else is "free".
(We prioritize leaders by the order we put the variables in when encoding our matrices. We *make a choice* and must stick with it for the whole calculation.)

For next time:

1. Moodle, syllabus, Slack, LaTeX, survey. . . (see "to do list" above)
2. Read Chapter one, Sections I, II, and III.
3. Read through "practice problems" and (time allowing) give some a try. (Never due, they're just for you. Solutions on textbook website.)
4. Some further questions to think on:
   (a) Are row operations "reversible"? i.e. if I use a row operation to get from matrix A to matrix B, can I get back to A again with a row operation? (Try to reverse each one!)
   (b) Given a fixed order for my variables, is echelon form unique? Is reduced echelon form unique? (Meaning, no matter what order I apply row operations, do I always get to the same answer?) *Hint:* Start by thinking about a system that has exactly one solution.