Recall: the definite integral $\int_a^b f(x)\,dx$ is the "signed" area under the curve $y = f(x)$. (If the curve is *above* the x-axis, you get a *positive* number; and if the curve is *below* the x-axis, you get a *negative* number.)

Let $f(x) = -x^2 + 5x - 6$.

1. Calculate the area between the x-axis and the curve $y = f(x)$ between $x = 1$ and $x = 2$.
2. Calculate the area of the region enclosed between the curve $y = -x^2 + 5x - 6$ and the x-axis.
3. Calculate the area contained between the curve $y = x^2 - 5x + 6$ and the x-axis.

Tip: Before you start, sketch $y = -(x^2 - 5x + 6)$. Also, all of your answers should be positive–we want *area* not "signed" area.)

Compute the following.

4. $\displaystyle\int x\sin(x^2)\,dx$

5. $\displaystyle\int x\sqrt{3-x}\,dx$

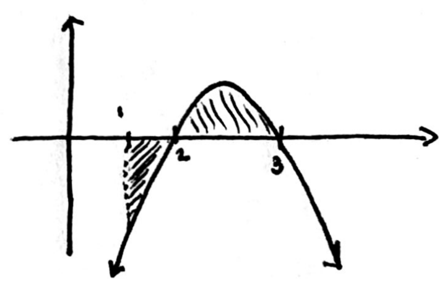
6. $\displaystyle\int \tan(x)\,dx$ [Hint: rewrite $\tan(x) = \frac{\sin(x)}{\cos(x)}$.]

1. $\text{Area} = -\int_1^2 f(x)\,dx$

$f(x) = -(x^2-5x+6)$
$= -(x-2)(x-3)$

$= -\int_1^2 -x^2+5x-6\,dx$

$= -\left(-\frac{x^3}{3} + \frac{5x^2}{2} - 6x + C\right)\Big|_1^2$



$= -\left(-\frac{8}{3} + \frac{20}{2} - 12 - \left(-\frac{1}{3} + \frac{5}{2} - 6\right)\right)$

$14 - 24 + 15$

$= -\left(-\frac{7}{3} + 4 - \frac{5}{2}\right) = 5/6$ (positive ✓)

2. $\text{Area} = \int_2^3 -x^2+5x-6\,dx = \left(-\frac{x^3}{3} + \frac{5x^2}{2} - 6x + c\right)\Big|_2^3$

$= -\frac{27}{3} + \frac{5\cdot 9}{2} - 6\cdot 3 - \left(-\frac{8}{3} + \frac{5\cdot 4}{2} - 6\cdot 2\right)$

$= 1/6$ (positive ✓)

3. Area = Area in #2 = 1/6 : =

4. $\int \sin(x^2) \cdot x\,dx$

let $u = x^2$, so $\frac{du}{dx} = 2x$. Thus $\frac{1}{2}du = x\,dx$.

$= \int \sin(u) \cdot \frac{1}{2}du \quad = \frac{1}{2}\int \sin(u)\,du$

$= \frac{1}{2}(-\cos(u)) + C$

$= -\frac{1}{2}\cos(x^2) + C.$

Check: $\frac{d}{dx} -\frac{1}{2}\cos(x^2) = -\frac{1}{2}(-\sin(x^2)) \cdot 2x = x\sin(x^2)$ ✓

5. $\int x\sqrt{3-x}\,dx$

let $u = 3-x$,

so $\frac{du}{dx} = -1$, giving $-du = dx$.

Also, solving for x gives

$x = 3-u$.

$= \int (3-u)\sqrt{u}\,(-du)$

$= -\int (3-u)u^{1/2}\,du \quad = -\int 3u^{1/2} - u^{3/2}\,du$

$= -3u^{3/2} \cdot \frac{2}{3} + u^{5/2} \cdot \frac{2}{5} + C$

$= -2(3-x)^{3/2} + \frac{2}{5}(3-x)^{5/2} + C$

Check: $\frac{d}{dx}\left(-2(3-x)^{3/2} + \frac{2}{5}(3-x)^{5/2} + C\right)$

$= -2 \cdot \frac{3}{2}(3-x)^{1/2}(-1) + \frac{2}{5} \cdot \frac{5}{2}(3-x)^{3/2}(-1) + 0$

$= +3\sqrt{3-x} - (3-x)\sqrt{3-x}$

$= (3-(3-x))\sqrt{3-x} = x\sqrt{3-x}$ ✓

6. $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{1}{\cos(x)} \cdot \sin(x)\,dx$

let $u = \cos(x)$, so $\frac{du}{dx} = -\sin(x)$.

Thus $-du = \sin(x)\,dx$.

$= \int \frac{1}{u}(-du) = -\int \frac{1}{u}\,du = -\ln|u| + C$

$= -\ln|\cos(x)| + C.$

Check: $\frac{d}{dx}\left(-\ln|\cos(x)| + C\right) = -\frac{1}{\cos(x)} \cdot (-\sin(x)) + 0$

$= \tan(x)$ ✓

You try:

Compute the following.

(a) $\int_{-1}^{7} (3x+1)^5 \, dx$

(b) $\int_{1}^{2} x\sqrt{3-x} \, dx$

(c) $\int_{\pi/4}^{\pi/3} \tan(x) \, dx$ [Hint: rewrite $\tan(x) = \frac{\sin(x)}{\cos(x)}$.]

Answers: (a) $\frac{1}{18}(22^6 - (-2)^6)$; (b) $\frac{-2}{5}(3(2)^{3/2} - 4)$; (c) $\frac{1}{2}\ln(2)$.

(a) $\int (3x+1)^5\,dx$ let $u = 3x+1$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$= \int u^5 \cdot \frac{1}{3}du = \frac{1}{3}\cdot\frac{u^6}{6} + C = \frac{1}{18}u^6 + C$

$= \frac{1}{18}(3x+1)^6 + C.$

So $\int_{-1}^{7}(3x+1)^5dx = \boxed{\frac{1}{18}\left(22^6 - 2^6\right).}$

(b) $\int_1^2 x\sqrt{3-x}\,dx = -2(3-x)^{3/2} + \frac{2}{5}(3-x)\Big|_1^2$

$= \boxed{-2\cdot 1^{3/2} + \frac{2}{5}\cdot 1 - \left(-2(2)^{3/2} + \frac{2}{5}(2)\right)}$

(c) $\int_{\pi/4}^{\pi/3}\tan(x)\,dx = -\ln|\cos(x)|\Big|_{\pi/4}^{\pi/3}$

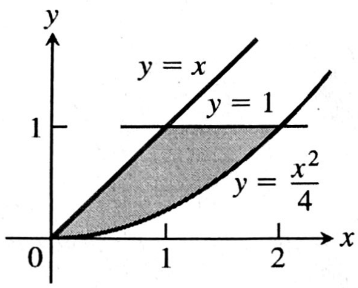
$= \boxed{-\ln|1/2| - \left(-\ln|1/\sqrt{2}|\right)}$

$= -\ln(1/2) + \ln(1/\sqrt{2}) = \ln(2) - \frac{1}{2}\ln(2) = \boxed{\frac{1}{2}\ln(2)}$

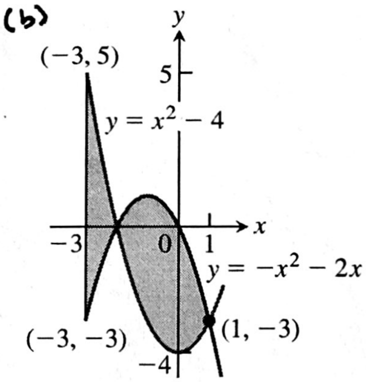
You try

1. Compute the following shaded areas.

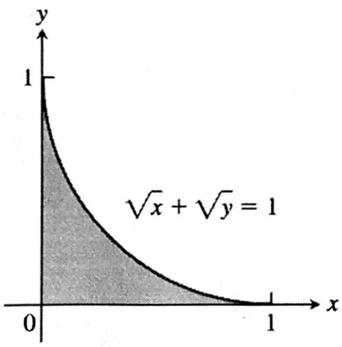
(a)



(b)



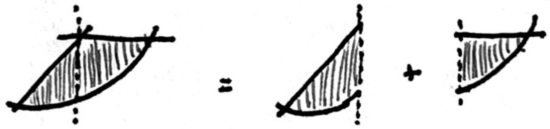
(c)



2. Draw the corresponding pictures and compute the areas of the described regions.
   (a) The region bounded by $y = x^2 - 2x$ and $y = 4 - x^2$.
   (b) The region(s) between $\sin x$ and $\cos x$ over $[-3\pi/4, 5\pi/4]$.
   (c) The region under the curve $y = \arcsin(x)$ over $[0, 1]$.

1. (a) Two methods: versus x or versus y.

Versus x: break into two pieces



$$= \int_0^1 x - \frac{x^2}{4}\,dx + \int_1^2 1 - \frac{x^2}{4}\,dx$$

$$= \frac{x^2}{2} - \frac{x^3}{12}\Big|_0^1 + \left(x - \frac{x^3}{12}\right)\Big|_1^2 = \frac{1}{2} - \frac{1}{12} + \left(2 - \frac{8}{12} - (1 - \tfrac{1}{12})\right)$$

$$= 2 - 9/12 = \boxed{5/6} \quad (>0 \checkmark)$$

– or –

Versus y: solve for curves versus y (solve for x)

$y = x \to \boxed{x = y}$

$y = x^2/4 \to 4y = x^2 \to \boxed{2\sqrt{y} = x}$

bands: $y = 0$ to $y = 1$

$$\int_0^1 2y^{1/2} - y\,dy = 2\cdot\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\Big|_0^1 = \frac{4}{3} - \frac{1}{2} = \boxed{5/6} \quad \text{(same as before} \checkmark)$$

(b) Area =  $\overset{*}{+}$  → find intersection points

$$x^2-4=-x^2-2$$

$$0=2x^2+2x-4$$

$$=2(x+2)(x-1)$$

$$\boxed{x=-2 \; *} \text{ and } x=1$$

$$=\int_{-3}^{-2} \underbrace{(x^2-4)-(-x^2-2x)}_{2x^2+2x-4}\,dx$$

$$+\int_{-2}^{1} \underbrace{(-x^2-2x)-(x^2-4)}_{-(2x^2+2x-4)}\,dx$$

$$=\frac{2x^3}{3}+x^2-4x\Big|_{-3}^{-2}+\left(-\left(\frac{2x^3}{3}+x^2-4x\right)\right)\Big|_{-2}^{1}$$

$$=\frac{2}{3}(-8)+4+8-\left(\frac{2}{3}(-27)+9+12\right)$$

$$-\left(\frac{2}{3}+1-4-\left(\frac{2}{3}(-8)+4+8\right)\right)=\frac{11}{3}+9=\boxed{38/3}$$

(c) Solve for either x or y:

Solve for x

$$\sqrt{x}=1-\sqrt{y}$$

$$x=(1-\sqrt{y})^2=1-2y^{1/2}+y$$

$$\int_0^1 1-2y^{1/2}+y\,dy$$

$$=\left(y-2\cdot\frac{2}{3}y^{3/2}+\frac{1}{2}y^2\right)\Big|_0^1$$

$$=1-\frac{4}{3}+\frac{1}{2}-0$$

$$=\boxed{\frac{1}{6}}$$

– or –

Solve for y

$$\sqrt{y}=1-\sqrt{x}$$

$$y=(1-\sqrt{x})^2$$

$$=1-2x^{1/2}+x$$

$$\int_0^1 1-2x^{1/2}+x\,dx$$

$$=\left(x-2\cdot\frac{2}{3}x^{3/2}+\frac{1}{2}x^2\right)\Big|_0^1$$

$$=1-\frac{4}{3}+\frac{1}{2}-0$$

$$=\boxed{1/6}$$ (same ✓)

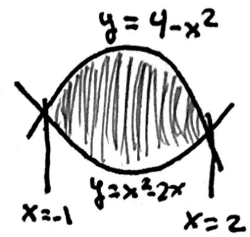
2(a) find intersection points:

$x^2-2x = 4-x^2$

$2x^2-2x-4=0$

$2(x-2)(x+1)=$

so $x=2$ or $x=-1$.

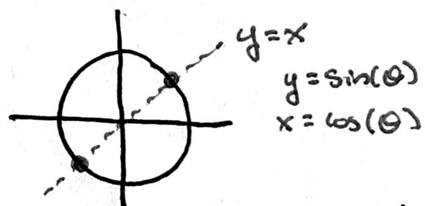


$$\text{Area} = \int_{-1}^{2} \underbrace{4-x^2-(x^2-2x)}_{4+2x-x^2}\,dx$$

$$= 4x+x^2-\frac{x^3}{3}\Big|_{-1}^{2} = 8+4-\frac{8}{3}-\left(-4+1+1/3\right)$$

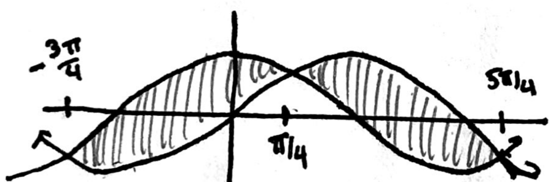
$$= 15-9/3 = \boxed{12}$$

(b) find intersection points:



$\theta = \frac{\pi}{4}, \frac{\pi}{4}+\cancel{\pi}, \frac{\pi}{4}+2\pi, \ldots$

$\frac{\pi}{4}-\pi, \frac{\pi}{4}-2\pi, \ldots$



intersection points in $\left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$:

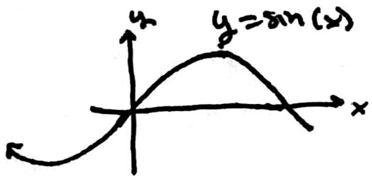
$$\boxed{-\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}}$$

$$\text{Area} = \int_{-\frac{3\pi}{4}}^{\pi/4} \cos(x)-\sin(x) + \int_{\pi/4}^{5\pi/4} \sin(x)-\cos(x)\,dx$$

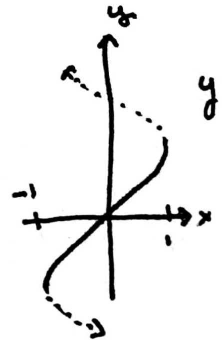
$$= \sin(x)+\cos(x)\Big|_{-3\pi/4}^{\pi/4} + \left(-(\cos(x)+\sin(x))\right)\Big|_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)-\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)\right) = \boxed{4\sqrt{2}}$$

(c) Recall

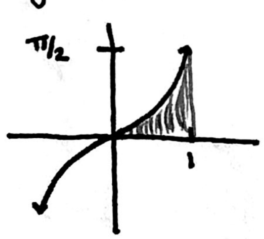


$\longrightarrow$



$y = \arcsin(x)$

$y = \arcsin(x)$



problem:

I don't know how to compute

$$\int \arcsin(x)\,dx$$

Switch to y:

$y = \arcsin(x) \longrightarrow x = \sin(y)$ (left function)

right function: $x = 1$

bounds: $y = 0$ to $y = \pi/2$.

$$\text{Area} = \int_0^{\pi/2} 1 - \sin(y)\,dy = y + \cos(y)\Big|_0^{\pi/2} = \frac{\pi}{2} + 0 - (0+1)$$

$$= \boxed{\frac{\pi}{2} - 1} \quad (>0 \checkmark)$$