Recall: the definite integral $\int_a^b f(x)\,dx$ is the "signed" area under the curve $y = f(x)$. (If the curve is *above* the x-axis, you get a *positive* number; and if the curve is *below* the x-axis, you get a *negative* number.)

Let $f(x) = -x^2 + 5x - 6$.

1. Calculate the area between the x-axis and the curve $y = f(x)$ between $x = 1$ and $x = 2$.
2. Calculate the area of the region enclosed between the curve $y = -x^2 + 5x - 6$ and the x-axis.
3. Calculate the area contained between the curve $y = x^2 - 5x + 6$ and the x-axis.

Tip: Before you start, sketch $y = -(x^2 - 5x + 6)$. Also, all of your answers should be positive–we want *area* not "signed" area.)

Compute the following.

4. $\displaystyle\int x\sin(x^2)\,dx$
5. $\displaystyle\int x\sqrt{3-x}\,dx$
6. $\displaystyle\int \tan(x)\,dx$ [Hint: rewrite $\tan(x) = \frac{\sin(x)}{\cos(x)}$.]

Quick note: Putting FTC and substitution together

Q. Calculate $\int_0^{\sqrt{\pi/2}} x\sin(x^2)\ dx$.

A. Separate your solution into two steps.

Step 1: Find the antiderivative $F(x)$ of $f(x) = x\sin(x^2)$.

Let $u = x^2$. So $du = 2x\ dx$, and $\frac{1}{2}\ du = x\ dx$.
Therefore

$$\int x\sin(x^2)\ dx = \int \sin(u) * \frac{1}{2}\ du$$

$$= -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2) + C$$

Step 2: Use your answer to compute

$$\int_0^{\sqrt{\pi/2}} x\sin(x^2)\ dx = F(\sqrt{\pi/2}) - F(0).$$

$$\int_0^{\pi/2} x\sin(x^2)\ dx = -\frac{1}{2}\cos((\sqrt{\pi/2})^2) - \left(-\frac{1}{2}\cos(0^2)\right) = 1/2$$

You try:

Compute the following.

(a) $\int_{-1}^{7} (3x+1)^5\ dx$

(b) $\int_1^2 x\sqrt{3-x}\ dx$

(c) $\int_{\pi/4}^{\pi/3} \tan(x)\ dx$ [Hint: rewrite $\tan(x) = \frac{\sin(x)}{\cos(x)}$.]

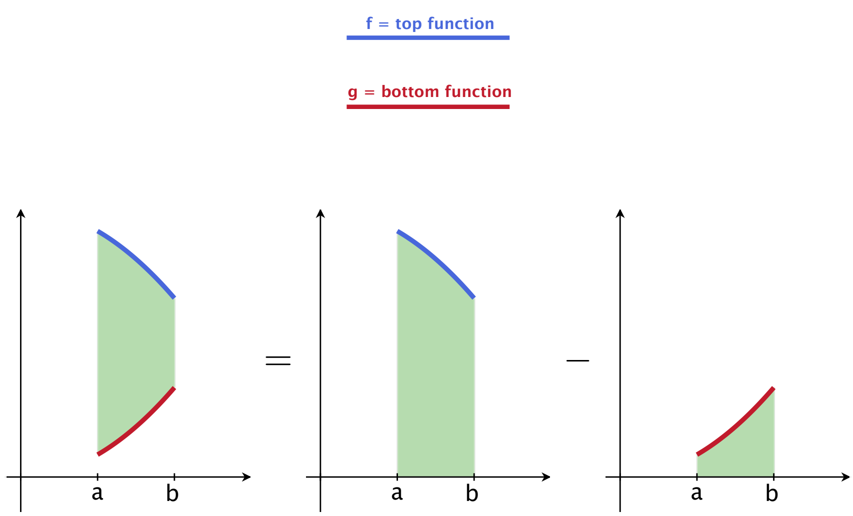
Answers: (a) $\frac{1}{18}(22^6 - (-2)^6)$; (b) $\frac{-2}{5}(3(2)^{3/2} - 4)$; (c) $\frac{1}{2}\ln(2)$.

Areas Between Curves

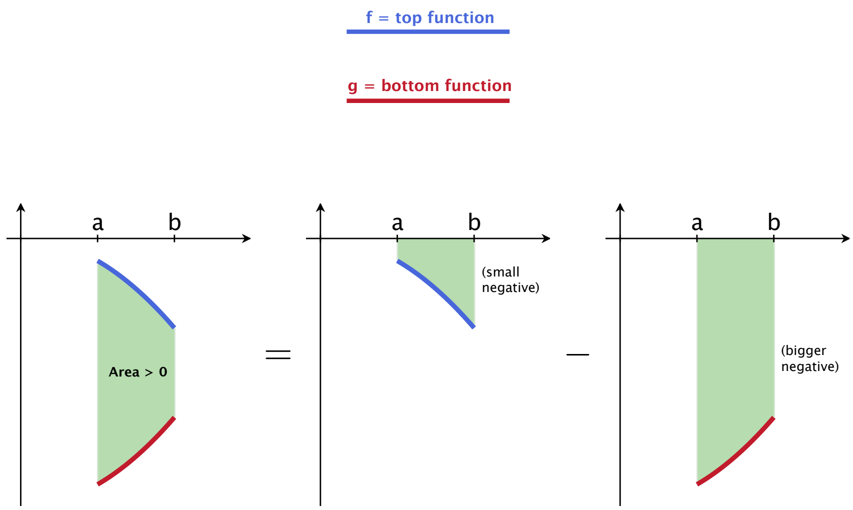
We know that if f is a continuous nonnegative function on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the area under the graph of f and above the interval.

Now suppose we are given two continuous functions, $f(x)$ and $g(x)$ so that $g(x) \leq f(x)$ for all x in the interval $[a, b]$.

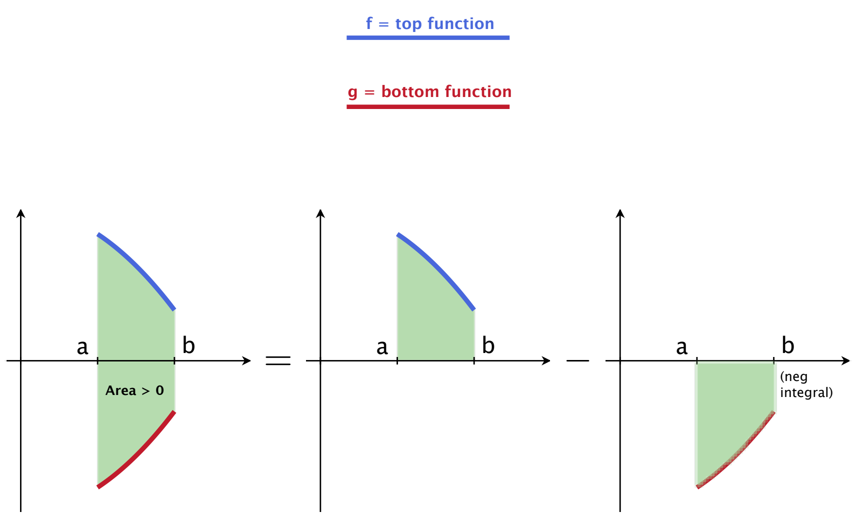
How do we find the area bounded by the two functions over that interval?



$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$

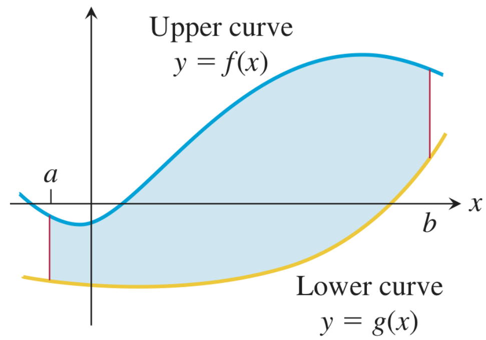


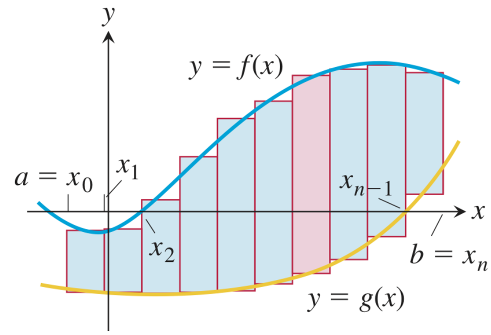
$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$



$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$

Looking back at Riemann sums:

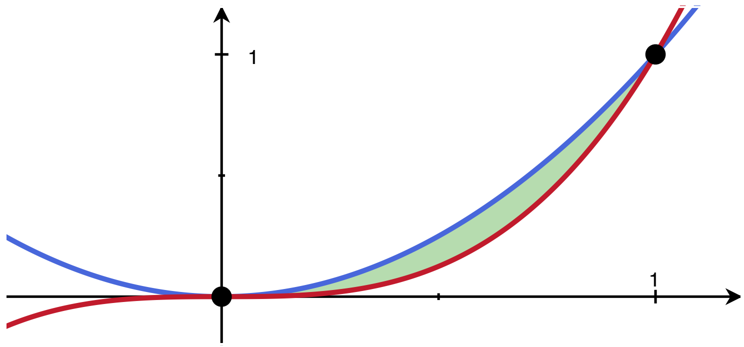




$$\int_a^b f(x) - g(x)dx \approx \sum_i \underbrace{(f(x_i) - g(x_i))}_{\text{height}} \underbrace{\Delta x}_{\text{width}}.$$

Example

Find the area of the region between the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$.



Top: x^2 Bottom: x^3

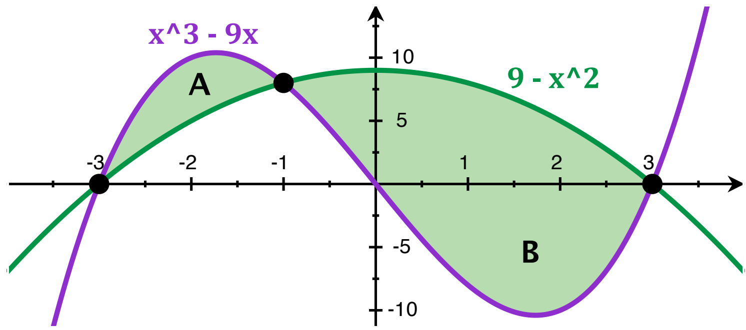
Intersections: where does $x^2 = x^3$? $\boxed{x = 0 \text{ or } 1}$

So Area $= \displaystyle\int_0^1 x^2 - x^3 dx = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_{x=0}^{1} = \boxed{\left(\tfrac{1}{3} - \tfrac{1}{4}\right) - 0} > 0$✓

Example

Find the area of the region bounded by the two curves $y = x^3 - 9x$ and $y = 9 - x^2$.

1. Check for intersection points (Solve $x^3 - 9x = 9 - x^2$).



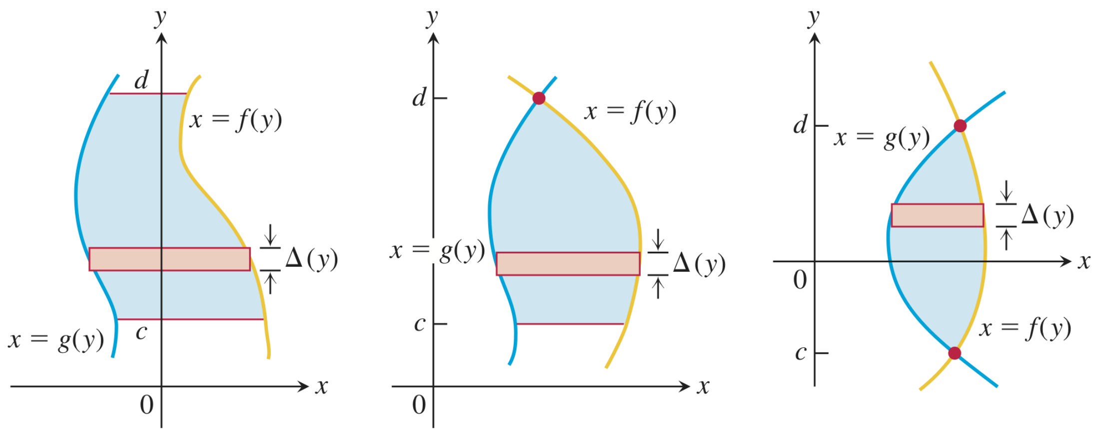
2. Area = Area A + Area B

Area A $= \int_{-3}^{-1} (x^3 - 9x) - (9 - x^2) dx = \int_{-3}^{-1} x^3 + x^2 - 9x - 9 \, dx$

Area B $= \int_{-1}^{3} (9 - x^2) - (x^3 - 9x) dx = -\int_{-1}^{3} x^3 + x^2 - 9x - 9 \, dx$

Functions of y

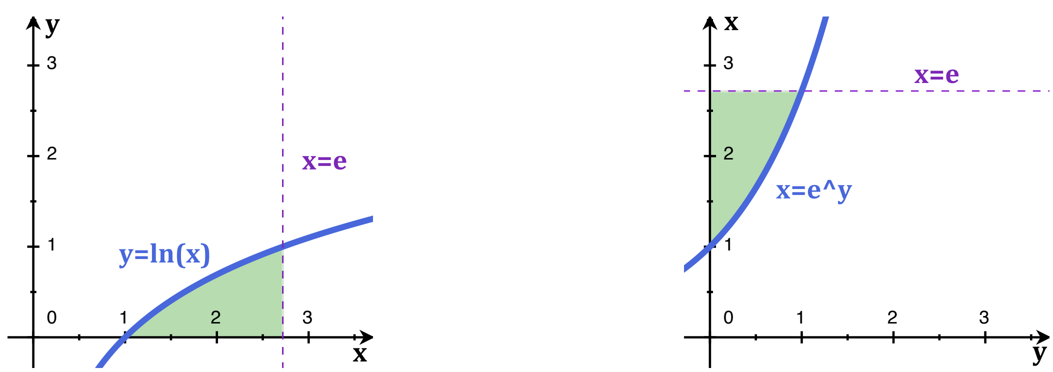
We could just as well consider two functions of y, say, $x = g_{Left}(y)$ and $x = f_{Right}(y)$ defined on the interval $[c, d]$.



$$\text{Area} = \int_c^d f(y) - g(y)\; dy.$$

Area Between the Two Curves

Find the area under the graph of $y = \ln x$ and above the interval $[1, e]$ on the x-axis.

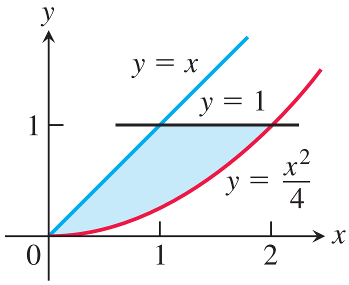


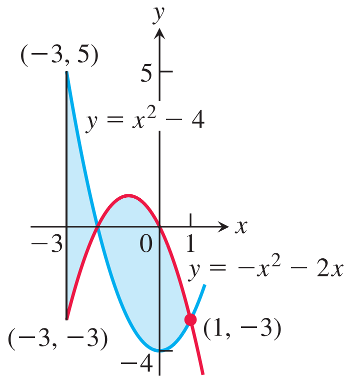
Uh oh...we don't know $\int \ln(x)\; dx$.

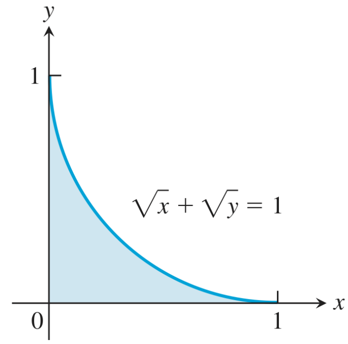
$$\text{Area} = \int_{y=0}^{1} e - e^y\; dy = (e * y - e^y)\big|_{y=0}^{1} = (e - e) - (0 - 1) = 1.$$

You try

1. Compute the following shaded areas.







2. Draw the corresponding pictures and compute the areas of the described regions.
   (a) The region bounded by $y = x^2 - 2x$ and $y = 4 - x^2$.
   (b) The region(s) between $\sin x$ and $\cos x$ over $[-3\pi/4, 5\pi/4]$.
   (c) The region under the curve $y = \arcsin(x)$ over $[0, 1]$.