Warmup

Fill in the blank:

1. Since $\dfrac{d}{dx}\cos(x^2+1) = \underline{\hspace{3cm}}$,

   so $\displaystyle\int \underline{\hspace{3cm}}\, dx = \cos(x^2+1) + C.$

2. Since $\dfrac{d}{dx}\ln|\cos(x)| = \dfrac{}{\underline{\hspace{2cm}}}$,

   so $\displaystyle\int \dfrac{}{\underline{\hspace{2cm}}}\, dx = \ln|\cos(x)| + C.$

(Example: $\frac{d}{dx}x^3dx = 3x^2$, so $\int 3x^2dx = x^3 + C$.)

Undoing chain rule

In general:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x),$$

so $$\int f'(g(x)) * g'(x)dx = f(g(x)) + C.$$

Example: Calculate the (extremely suggestively written) integral

$$\int \cos(x^3+5x-10) * (3x^2 + 5 * 1 + 0)dx$$

Less obvious chain rules.

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx:

Examples:

$$\int \frac{\cos(x)}{\sin(x)+1} dx = \int \frac{1}{\sin(x)+1} * \cos(x)\ dx$$

$$\int x\sqrt{x^2+1} dx = \int \frac{1}{2}\sqrt{x^2+1} * 2x\ dx$$

$$\int \frac{\cos(\sqrt{x})}{2\sqrt{x}} dx = \int \cos(\sqrt{x}) * \frac{1}{2\sqrt{x}}\ dx$$

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $\int \frac{\cos(x)}{\sin(x)+1} dx$

Let $u = g(x)$.	Let $u = \sin(x) + 1$
Calculate du.	$\frac{du}{dx} = \cos(x)$ so $du = \cos(x)dx$
Clear out all of the x's, replacing them with u's.	$\int \frac{1}{u} du$
Calculate the new integral.	$\int \frac{1}{u} du = \ln\|u\| + C$
Substitute back into x's.	$\ln\|u\| + C = \ln\|\sin(x)+1\| + C$

Check $\frac{d}{dx} \ln|\sin(x)+1| + C = \frac{1}{\sin(x)+1} * \cos(x)$✓

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $$\int x\sqrt{x^2+1}\,dx$$

Let $u = g(x)$.	Let $u = x^2+1$
Calculate $c * du$.	$\frac{du}{dx} = 2x$ so $\frac{1}{2}du = x\ dx$
Clear out all of the x's, replacing them with u's.	$\int \sqrt{u} * \frac{1}{2}du$
Calculate the new integral.	$\frac{1}{2}\int u^{1/2}du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$
Substitute back into x's.	$= \frac{1}{3}(x^2+1)^{3/2} + C$

Check $\frac{d}{dx}\frac{1}{3}(x^2+1)^{3/2} + C = \frac{1}{3}\frac{3}{2}(x^2+1)^{1/2} * 2x$✓

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

Let $u = g(x)$.	Let $u = \sqrt{x}$
Calculate $c * du$.	$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ so $2du = \frac{1}{\sqrt{x}}\ dx$
Clear out all of the x's, replacing them with u's.	$\int e^u * 2du$
Calculate the new integral.	$2\int e^u du = 2e^u + C$
Substitute back into x's.	$= 2e^{\sqrt{x}} + C$

Check $\frac{d}{dx}2e^{\sqrt{x}} + C = 2e^{\sqrt{x}} * \frac{1}{2}\frac{1}{\sqrt{x}}$✓

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $\displaystyle\int \frac{2x}{\sqrt[3]{x^2+1}}dx$

Let $u = g(x)$. Let $u = x^2 + 1$

Calculate $c * du$. $\frac{du}{dx} = 2x$ so $du = 2x\ dx$

Clear out all of the x's, replacing them with u's. $\displaystyle\int \frac{1}{\sqrt[3]{u}}du = \int u^{-1/3}du$

Calculate the new integral. $\int u^{-1/3}du = \frac{3}{2}u^{2/3} + C$

Substitute back into x's. $= \frac{3}{2}(x^2+1)^{2/3} + C$

Check $\frac{d}{dx}\frac{3}{2}(x^2+1)^{2/3} + C = \frac{3}{2}\frac{2}{3}(x^2+1)^{-1/3}2x\checkmark$

Same integral, different substitution!

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $\displaystyle\int \frac{2x}{\sqrt[3]{x^2+1}}dx$

Let $u = g(x)$. Let $u = \sqrt[3]{x^2+1}$

Calculate $c * du$. $\frac{du}{dx} = \frac{1}{3}(x^2+1)^{-2/3}2x$

Uh oh? Let's try to force it!! Cross-multiply:

$$2xdx = \frac{du}{\frac{1}{3}(x^2+1)^{-2/3}} = 3(x^2+1)^{2/3}du = 3u^2du.$$

Yay!! So

$$\int \frac{2x}{\sqrt[3]{x^2+1}}dx = \int \frac{3u^2}{u}du = 3\int u\ du = \frac{3}{2}u^2 + C$$

$$= \frac{3}{2}(\sqrt[3]{x^2+1})^2 + C = \frac{3}{2}(x^2+1)^{2/3} + C,$$

just like before!

You try: Compute the following using substitution. Check your answer each time by taking a derivative.

1. $\int (3x+7)^5 \, dx$
2. $\int \sqrt{5x-9} \, dx$
3. $\int \frac{1}{\sqrt{4x+3}} \, dx$
4. $\int \frac{1}{\sqrt{3-4x}} \, dx$
5. $\int \frac{x+1}{x^2+2x-3} \, dx$
6. $\int \frac{4x-5}{2x^2-5x+1} \, dx$
7. $\int \frac{2x+3}{\sqrt{x^2+3x-2}} \, dx$
8. $\int \frac{dx}{\sqrt{1-3x}-\sqrt{5-3x}}$
9. $\int \frac{x^2}{1+x^6} \, dx$
10. $\int (1-x)\sqrt{1+x} \, dx$
11. $\int \sin 3x \, dx$
12. $\int \csc^2(2x+5) \, dx$
13. $\int \sin x \cos x \, dx$
14. $\int \sin^3 x \cos x \, dx$
15. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx$