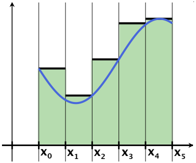
Recall from last time, that the definite integral of a function f over an interval $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where

1. $\Delta x = \frac{b-a}{n}$,
2. $x_i = a + i\Delta x$, and
3. c_i is any point in the interval $[x_{i-1}, x_i]$.

To compute, set up a finite Reimann sum



and then take the limit as the number of subdivisions goes to ∞.

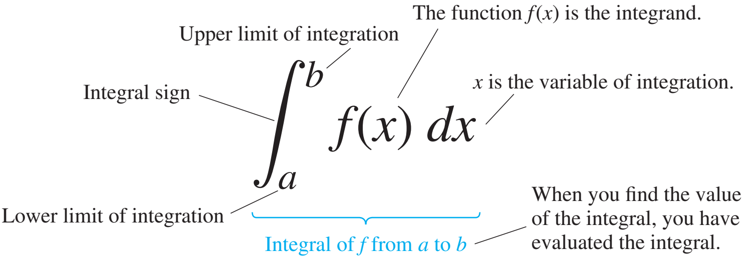
Warmup: Set up the limit definition of $\displaystyle\int_{-1}^{5} \sin(x)dx$, using the midpoints of each interval (picking $c_i = \frac{1}{2}(x_i - x_{i-1})$).

Vocabulary:

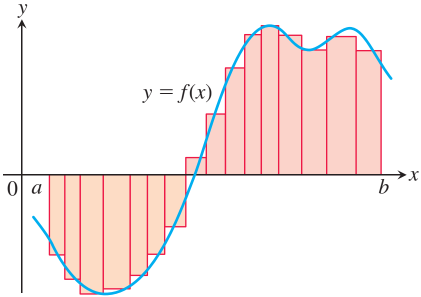
Reimann sum: $\sum_{i=1}^{n} f(c_i)\Delta x$ Upper sum: Choose c_i so that $f(c_i)$ is maximal over $[x_{i-1}, x_i]$ (overestimate).

Lower sum: Choose c_i so that $f(c_i)$ is minimal over $[x_{i-1}, x_i]$ (underestimate).

Midpoint rule: Choose c_i halfway between x_{i-1} and x_i.

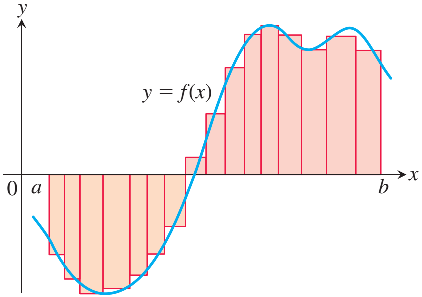


Properties of the Definite Integral



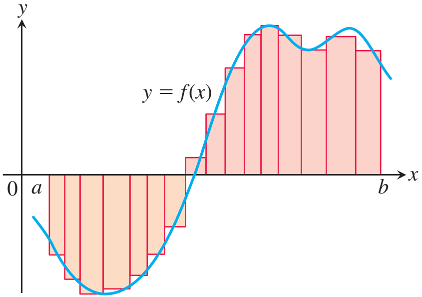
1. $\displaystyle\int_a^a f(x)\, dx = 0.$

Properties of the Definite Integral



1. $\displaystyle\int_a^a f(x)\ dx = 0.$
2. If f is integrable and
   (a) $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\ dx$ equals the area of the region under the graph of f and above the interval $[a, b]$;
   (b) $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)\ dx$ equals the **negative** of the area of the region between the interval $[a, b]$ and the graph of f.

Properties of the Definite Integral



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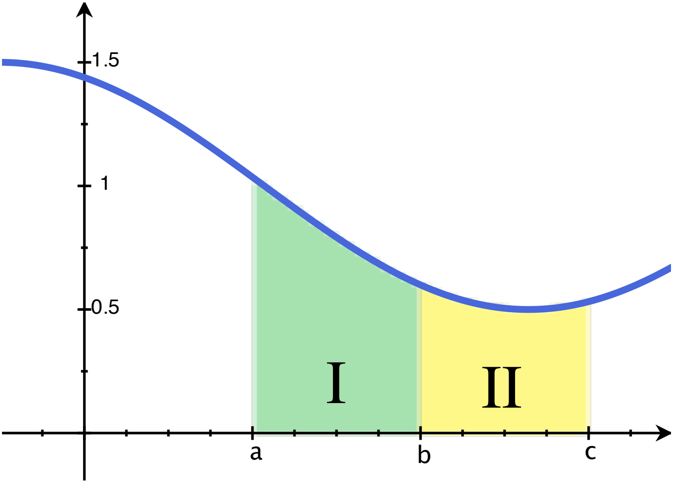
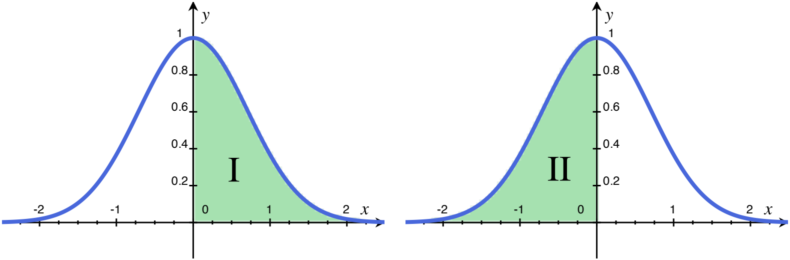
   (b) $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ equals the **negative** of the area of the region between the interval $[a, b]$ and the graph of f.
3. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

4. If $a < b < c$, $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$



5. If f is an **even** function, then

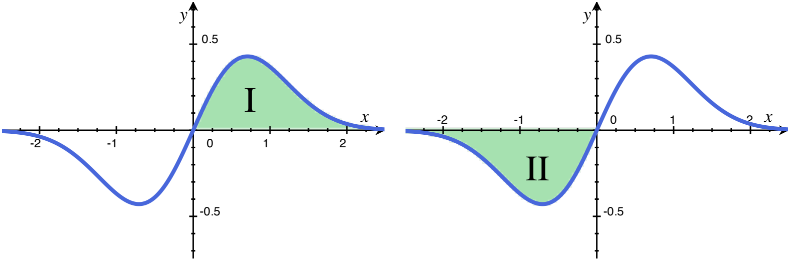
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



Area I = Area II

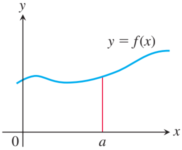
6. If f is an **odd** function, then

$$\int_{-a}^{a} f(x)dx = 0.$$



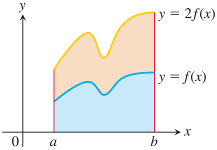
Area I = Area II

See book:



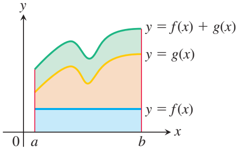
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



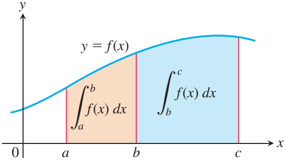
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



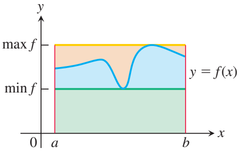
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



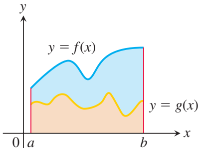
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$(\min f)\cdot(b - a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b - a)$$



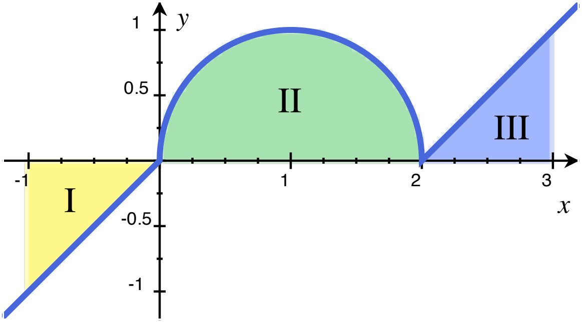
(f) *Domination:*

If $f(x) \ge g(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

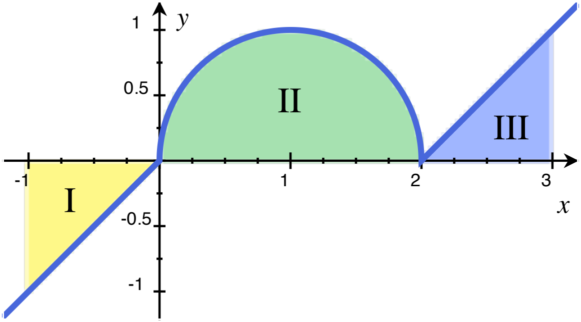
Example

$$\text{If } f(x) = \begin{cases} x, & x < 0, \\ \sqrt{1-(x-1)^2}, & 0 \geq x \leq 2, \\ x-2, & x \geq 2, \end{cases} \text{ what is } \int_{-1}^{3} f(x)dx?$$



Example

If $f(x) = \begin{cases} x, & x < 0, \\ \sqrt{1-(x-1)^2}, & 0 \geq x \leq 2, \\ x-2, & x \geq 2, \end{cases}$ what is $\displaystyle\int_{-1}^{3} f(x)dx$?



You try: Show $1 \leq \int_0^1 \sqrt{1+\cos(x)}\,dx \leq \sqrt{2}$.

Mean Value Theorem for Definite Integrals

Theorem

Let f be continuous on the interval $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c).$$

Compare to the mean value theorem from before!

Mean Value Theorem for Definite Integrals

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Let f be continuous on the interval $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c).$$

Compare to the mean value theorem from before!

Definition

The average value of a continuous function on the interval $[a, b]$ is

$$\frac{1}{b - a}\int_a^b f(x)dx.$$

Warm-up

Suppose a particle is traveling at velocity $v(t) = t^2$ from $t = 1$ to $t = 2$. if the particle starts at $y(0) = y_0$,

1. what is the function $y(t)$ which gives the particles position as a function of time (will have a y_0 in it)?
2. how far does the particle travel from $t = 1$ to $t = 2$?

Compare your answer to the upper and lower estimates of the area under the curve $f(x) = x^2$ from $x = 1$ to $x = 2$:

Upper

$$\sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right)$$

Lower

$$\sum_{i=0}^{n-1} \left(1 + \frac{i}{n}\right)^2 * \left(\frac{1}{n}\right)$$

n	Upper	Lower
10	2.485	2.185
100	2.34835	2.31835
1000	2.33483	2.33183

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The Fundamental Theorem of Calculus

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$$\int_a^b f(x)dx = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3} + C$

So

$$\int_1^2 x^2 dx = F(2) - F(1) = \left(\frac{2^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right)$$

$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333}$$

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The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ *is any function satisfying* $\frac{d}{dx}F(x) = f(x)$, *then*

$$\int_a^b f(x)dx = F(x)\Big|_{x=a}^{b} = F(b) - F(a)$$

Q. What is $\displaystyle\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3}$
So

$$\int_1^2 x^2 dx = \frac{x^3}{3}\Bigg|_{x=1}^{2} = \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right)$$

$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333}$$ (same answer!)

Examples

Use the fundamental theorem of calculus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

to calculate

1. $\int_2^3 3x \ dx$
2. $\int_{-1}^1 x^3 \ dx$
3. $\int_0^\pi \sin(x) \ dx$
4. $\int_\pi^0 \sin(x) \ dx$

Examples

Use the fundamental theorem of calculus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

to calculate

1. $\displaystyle\int_2^3 3x\ dx = 3\frac{x^2}{2}\Big|_{x=4}^{6} = 3 * \frac{9}{2} - 3 * \frac{4}{2} = 15/2$
2. $\displaystyle\int_{-1}^1 x^3\ dx = \frac{x^4}{4}\Big|_{x=-1}^{1} = \frac{1^4}{4} - \frac{(-1)^4}{4} = 0$ (odd function!!)
3. $\displaystyle\int_0^\pi \sin(x)\ dx = -\cos(x)\Big|_{x=0}^{\pi} = -\cos(\pi) - (-\cos(0))$
   $= -(-1) - (-1) = 2$
4. $\displaystyle\int_\pi^0 \sin(x)\ dx = -\cos(x)\Big|_{x=\pi}^{0} = -\cos(0) - (-\cos(\pi))$
   $= -(1) - (-(-1)) = -2$

The Fundamental Theorem of Calculus

Theorem (the big case)

If $F(x)$ is any function satisfying $\frac{d}{dt}F(t) = f(t)$, then

$$\int_{a(x)}^{b(x)} f(t)dt = F(t)\Big|_{t=a(x)}^{b(x)} = F(b(x)) - F(a(x))$$

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Theorem (the big case)

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Q. What is $\displaystyle\int_{\sin(x)}^{\ln(x)} t^2 dt$?

The Fundamental Theorem of Calculus

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Q. What is $\displaystyle\int_{\sin(x)}^{\ln(x)} t^2 dt$?

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The Fundamental Theorem of Calculus

Theorem (the big case)

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Q. What is $\displaystyle\int_{\sin(x)}^{\ln(x)} t^2 dt$?

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So

$$\int_{\sin(x)}^{\ln(x)} t^2 dt = \frac{1}{3}t^3\Bigg|_{t=\sin(x)}^{\ln(x)} = \left(\frac{1}{3}(\ln(x))^3\right) - \left(\frac{1}{3}(\sin(x))^3\right).$$

Examples

Use the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(t)dt = F(b(x)) - F(a(x))$$

to calculate

1. $\displaystyle\int_{\sin(x)}^{\cos(x)} 3t \; dt$
2. $\displaystyle\int_{x+1}^{5x^2-3} t^3 \; dt$
3. $\displaystyle\int_{\arccos(x)}^{0} \sin(t) \; dt$

Examples

Use the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(t)dt = F(b(x)) - F(a(x))$$

to calculate

1. $\displaystyle\int_{\sin(x)}^{\cos(x)} 3t\ dt\ = \frac{3}{2}t^2\Big|_{t=\sin(x)}^{\cos(x)} = \frac{3}{2}(\cos(x))^2 - \frac{3}{2}(\sin(x))^2$
2. $\displaystyle\int_{x+1}^{5x^2-3} t^3\ dt\ = \frac{1}{4}t^4\Big|_{t=x+1}^{5x^2-3} = \frac{1}{4}(5x^2-3)^4 - \frac{1}{4}(x+1)^4$
3. $\displaystyle\int_{\arccos(x)}^{0} \sin(t)\ dt\ = -\cos(t)\Big|_{t=\arccos(x)}^{0}$

   $= -\cos(0) - (-\cos(\arccos(x))) = -(1) - (-(x)) = x - 1$

For reference, we calculated $\int_{a(x)}^{b(x)} f(t)\ dt$ where

$$f(t) = t^2 \qquad a(x) = \sin(x) \qquad b(x) = \ln(x).$$

Notice:

$$\left(\frac{1}{3}(\ln(x))^3 - \frac{1}{3}(\sin(x))^3\right)$$

For reference, we calculated $\displaystyle\int_{a(x)}^{b(x)} f(t)\ dt$ where

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Notice:

$$\frac{d}{dx}\left(\frac{1}{3}(\ln(x))^3 - \frac{1}{3}(\sin(x))^3\right) = \frac{1}{x}(\ln(x))^2 - \cos(x)(\sin(x))^2$$

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In general:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\, dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

(Don't even have to know $F(t)$!)

Why?

Example: Calculate $\dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt$.

Why?

Example: Calculate $\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt$.

Answer: We can't even calculate $\int e^{t^2}\ dt$!

(There is no elementary function $F(t)$ which satisfies $F'(t) = e^{t^2}$)

But we know $\int e^{t^2}\ dt$ is a function. Call it $F(t)$.

Why?

Example: Calculate $\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt$.

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So $\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt = F(\sin(x)) - F(\tan(x))$.

Why?

Example: Calculate $\dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt$.

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Therefore $\quad \dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt$

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Therefore $$\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt = \frac{d}{dx}\left(F(\sin(x)) - F(\tan(x))\right)$$

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Therefore
$$\begin{aligned}\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt &= \frac{d}{dx}\left(F(\sin(x)) - F(\tan(x))\right)\\ &= \cos(x)F'(\sin(x)) - \sec^2(x)F'(\tan(x))\end{aligned}$$

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Example: Calculate $\dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt$.

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Therefore

$$\begin{aligned}
\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt &= \frac{d}{dx}\left(F(\sin(x)) - F(\tan(x))\right) \\
&= \cos(x)F'(\sin(x)) - \sec^2(x)F'(\tan(x)) \\
&= \cos(x)f(\sin(x)) - \sec^2(x)f(\tan(x))
\end{aligned}$$

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Example: Calculate $\dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\ dt$.

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