The purpose of calculus is twofold:

1. to find how something is changing, given what it's doing;
2. to find what something is doing, given how it's changing.

We did derivatives

(a) **algebraically** (derivative rules, what is the function?), and

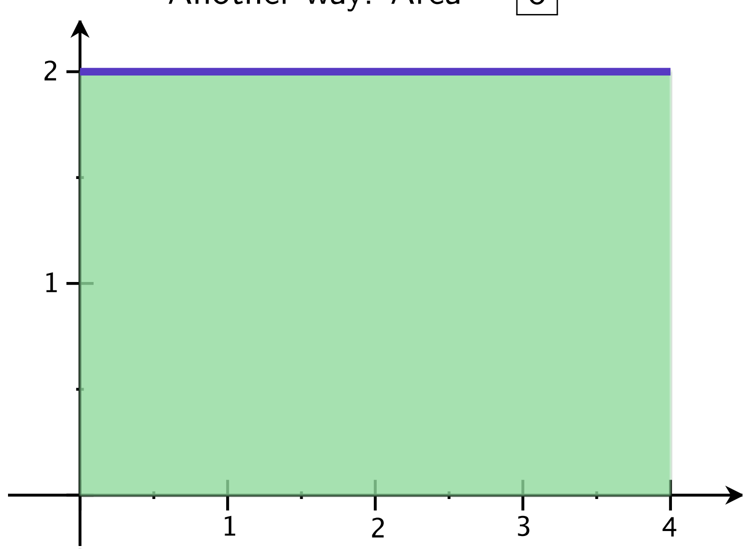
(b) **geometrically** (slopes, increasing/decreasing, what does it look like?)

We did antiderivatives algebraically (what is the function?).
Today: geometric meaning of antiderivatives.

If you travel at 2 mph for 4 hours, how far have you gone?

Answer: 8 miles.

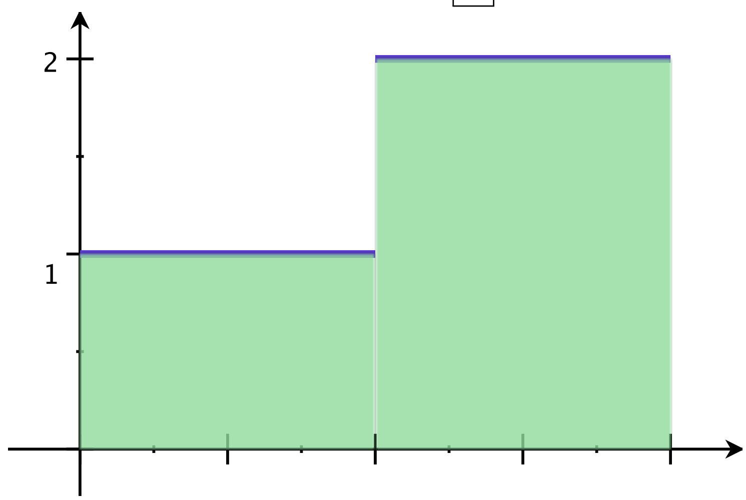
Another way: Area = $\boxed{8}$



(graph of speed, i.e. graph of derivative)

If you travel at 1 mph for 2 hours, and 2 mph for 2 hours, how far have you gone?

Area = 2+4= $\boxed{6}$



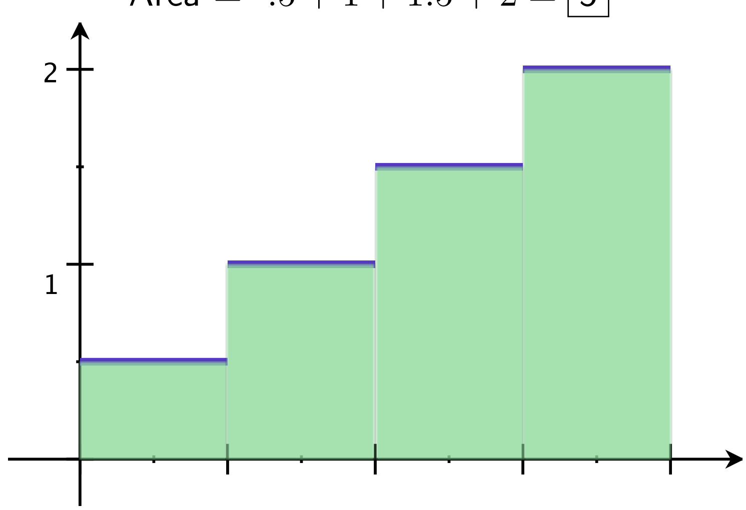
(graph of speed, i.e. graph of derivative)

If you travel at

.5 mph for 1 hour,
1 mph for 1 hour,
1.5 mph for 1 hour,
2 mph for 1 hour,

how far have you gone?

Area = $.5 + 1 + 1.5 + 2 = \boxed{5}$



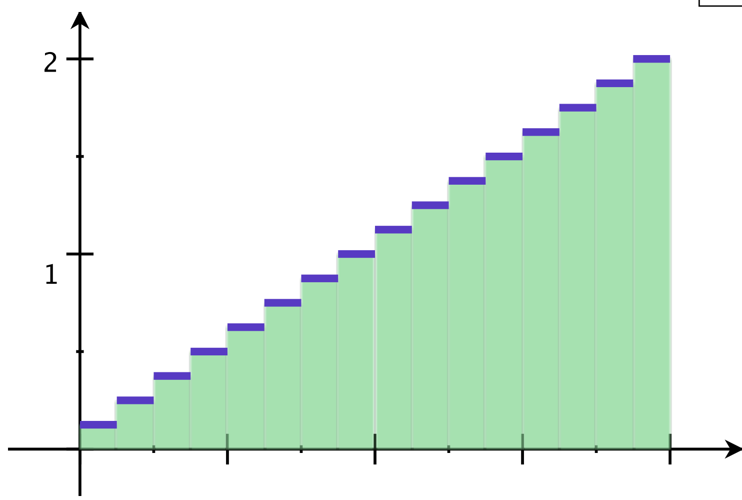
(graph of speed, i.e. graph of derivative)

If you travel at

.175 mph for 1/4 hour,
.25 mph for 1/4 hour,
...
2 mph for 1/4 hour,

how far have you gone?

$$\text{Area} = .175 * .25 + .25 * .25 + \cdots + 2 * .25 = \boxed{4.25}$$



(graph of speed, i.e. graph of derivative)

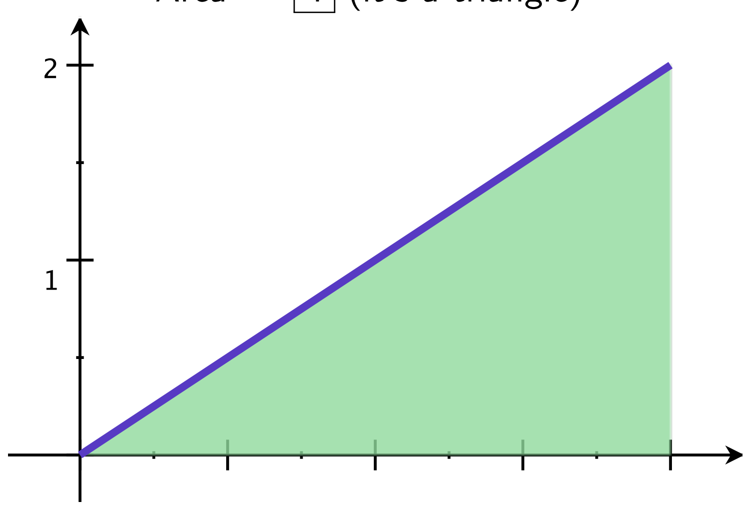
If you travel at $\frac{1}{2}t$ mph for 4 hours, how far have you gone?

Check our answer using antiderivatives from last time:

$$\text{position} = s(t) = \int \frac{1}{2}t\ dt = \frac{1}{4}t^2 + C$$

So distance $= s(4) - s(0) = \frac{1}{4} * 16 + C - (\frac{1}{4} * 0 + C) = 4$ ✓

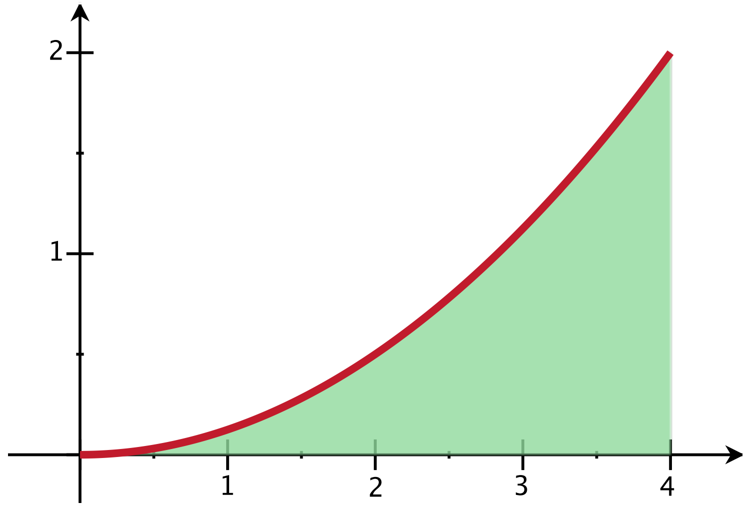
$$\text{Area} = \boxed{4} \text{ (it's a triangle)}$$



(graph of speed, i.e. graph of derivative)

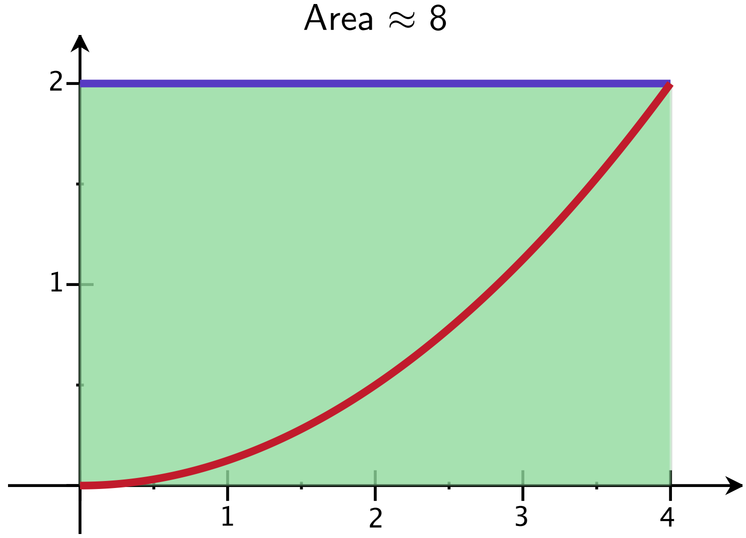
Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Area = ???



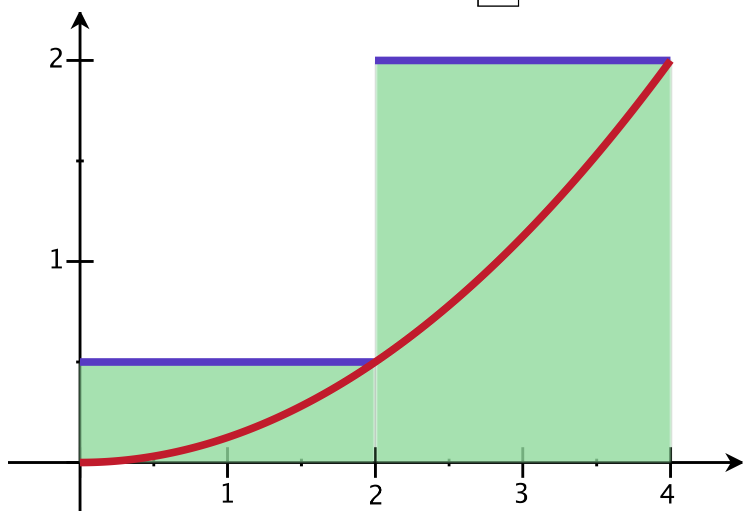
Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 1: pick the highest point
Area $\approx$ 8



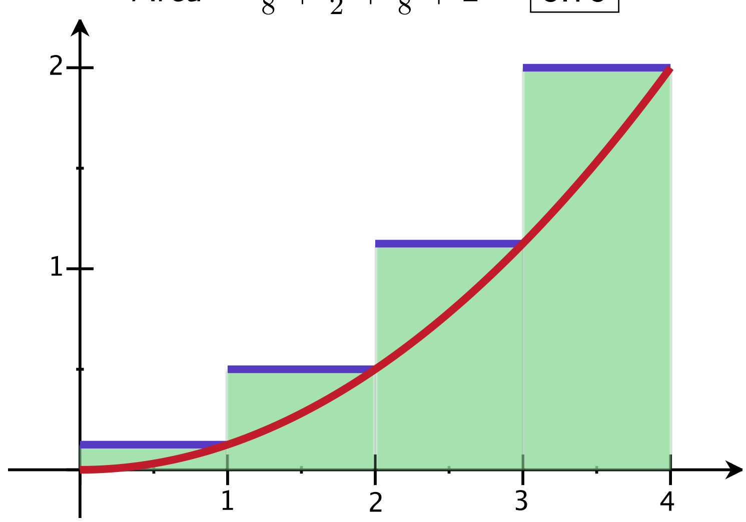
Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 2: pick two points
Area $\approx$ 1+4 = $\boxed{5}$



Estimate the area under the curve
$y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

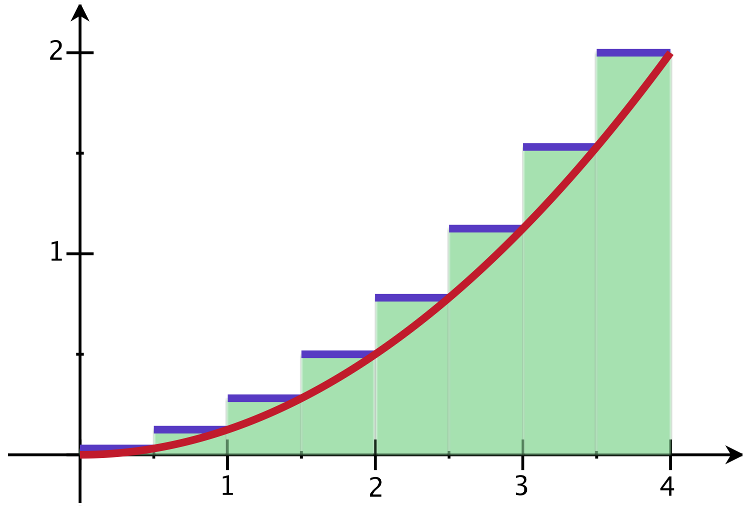
Estimate 3: pick four points
Area $\approx \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 = \boxed{3.75}$



Estimate the area under the curve $y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 4: pick eight points

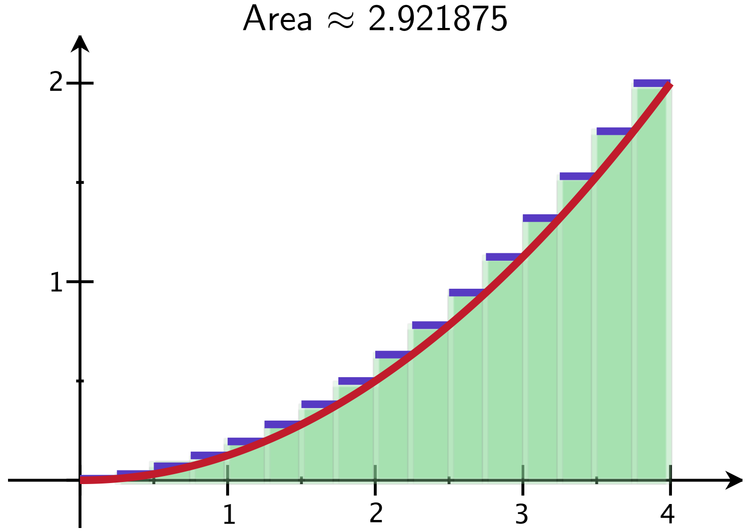
Area $\approx \frac{1}{32} * \frac{1}{2} + \frac{1}{8} * \frac{1}{2} + \cdots + 2 * \frac{1}{2} = \boxed{3.1875}$



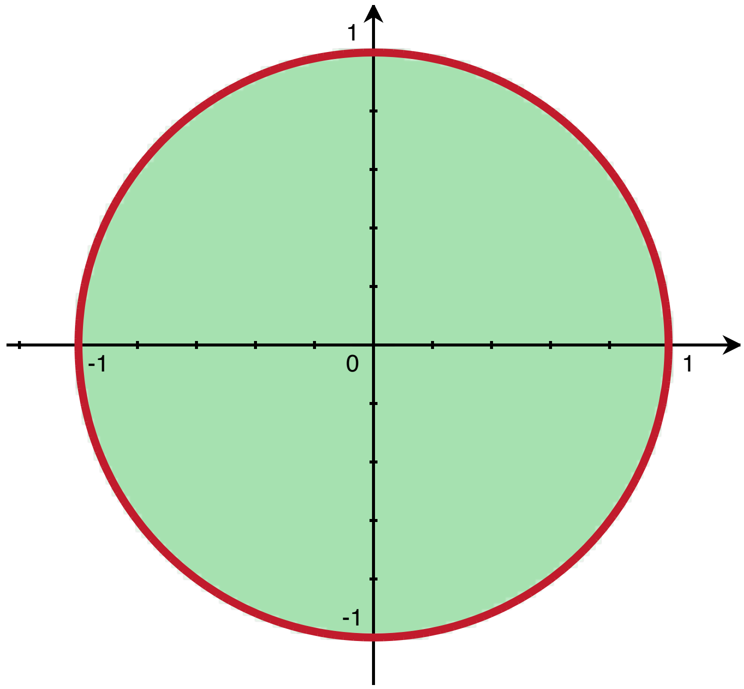
Estimate the area under the curve $y = \frac{1}{8}x^2$ between $x = 0$ and $x = 4$:

Estimate 5: pick sixteen points

Area ≈ 2.921875

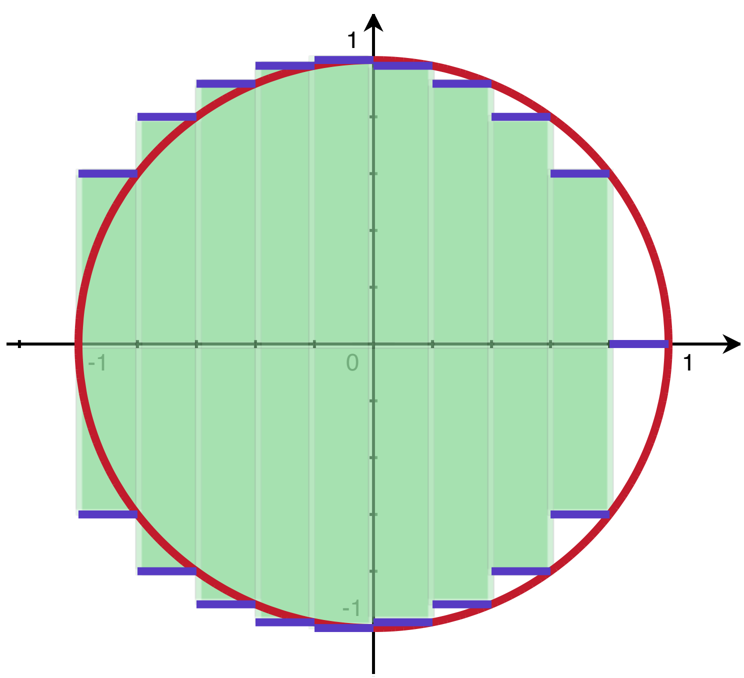


Estimating the Area of a Circle with $r = 1$



Estimating the Area of a Circle with $r = 1$

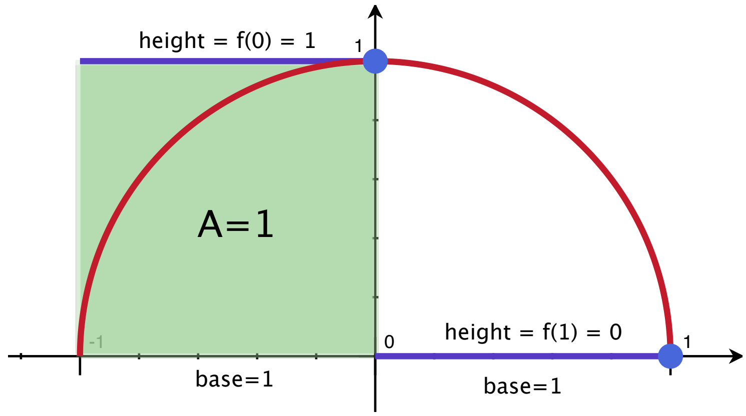
Divide it up into rectangles:



Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:

Estimate area of the half circle ($f(x) = \sqrt{1 - x^2}$) and mult. by 2.



# rect.	Area
4	2*1 = 2
4*2	
4*3	
4*4	
4*5	

Estimating the Area of a Circle with $r = 1$

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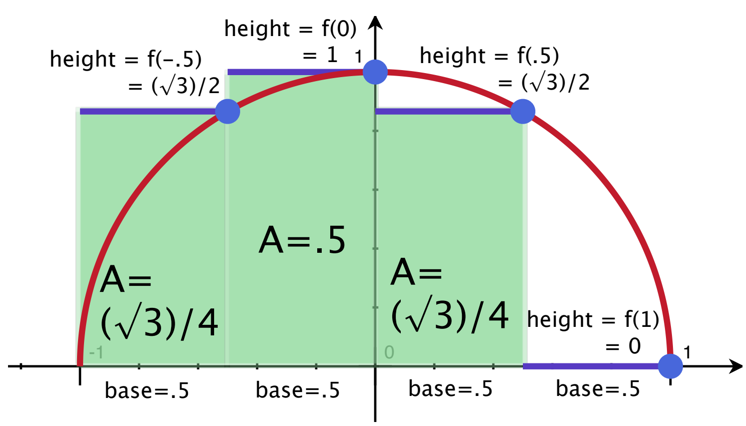
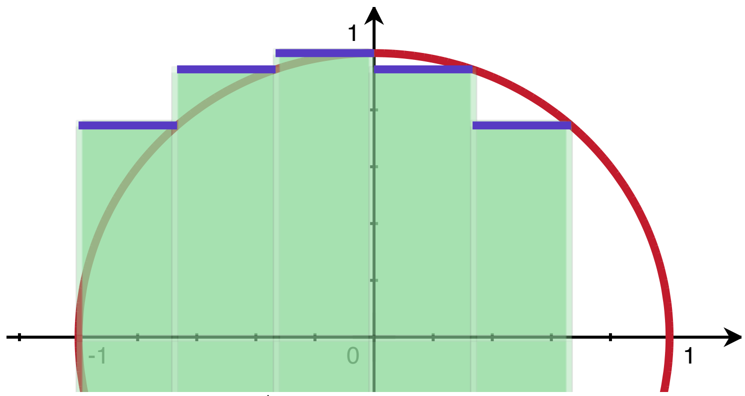
Estimate area of the half circle ($f(x) = \sqrt{1 - x^2}$) and mult. by 2.



# rect.	Area
4	2*1 = 2
4*2	$\sqrt{3} + 1 \approx 2.732$
4*3	
4*4	
4*5	

Estimating the Area of a Circle with $r = 1$

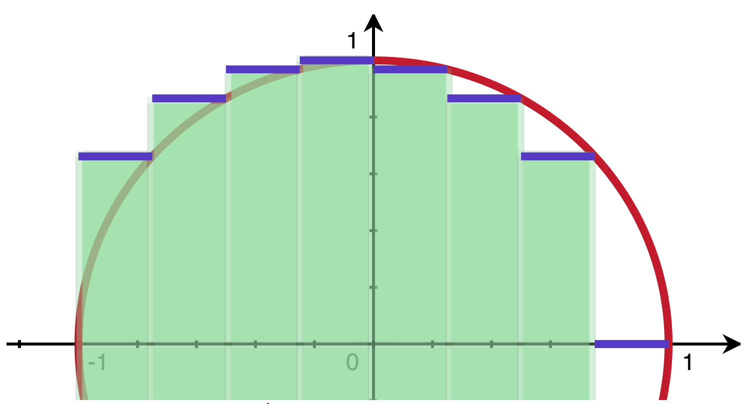
Divide it up into rectangles:
Estimate area of the half circle ($f(x) = \sqrt{1-x^2}$) and mult. by 2.



# rect.	Area
4	2*1 = 2
4*2	$\sqrt{3}+1 \approx 2.732$
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	
4*5	

Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:
Estimate area of the half circle ($f(x) = \sqrt{1-x^2}$) and mult. by 2.

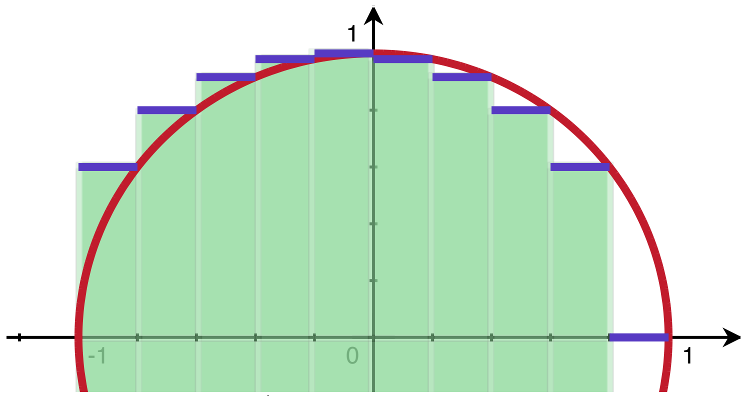


# rect.	Area
4	2*1 = 2
4*2	$\sqrt{3}+1 \approx 2.732$
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	2.996
4*5	

Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:

Estimate area of the half circle ($f(x) = \sqrt{1 - x^2}$) and mult. by 2.

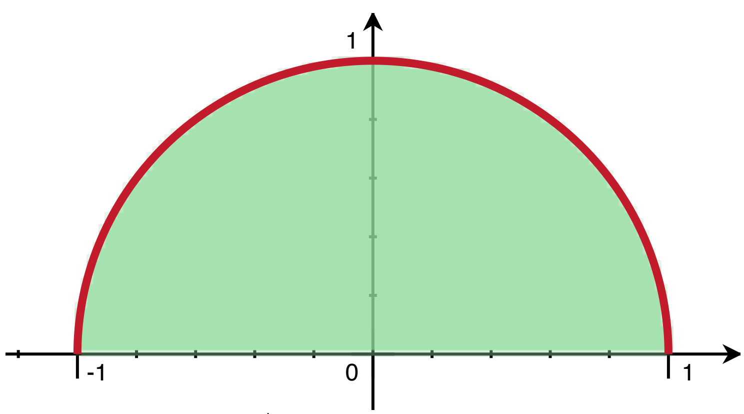


# rect.	Area
4	2*1 $= 2$
4*2	$\sqrt{3} + 1 \approx$ 2.732
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	2.996
4*5	3.037

Estimating the Area of a Circle with $r = 1$

Divide it up into rectangles:

Estimate area of the half circle ($f(x) = \sqrt{1 - x^2}$) and mult. by 2.



# rect.	Area
4	2*1 $= 2$
4*2	$\sqrt{3} + 1 \approx$ 2.732
4*3	$\frac{2}{3} + \frac{8\sqrt{2}+4\sqrt{5}}{9} \approx 2.918$
4*4	2.996
4*5	3.037
4*100	3.140

Numerical Integration

Big idea: Estimating, and then taking a limit.

Let the number of pieces go to ∞
i.e. let the base of the rectangle for to 0.

Good for:

1. Approximating accumulated change when the antiderivative is unavailable.
2. Making precise the notion of 'area' (we'll also to lengths and volumes)

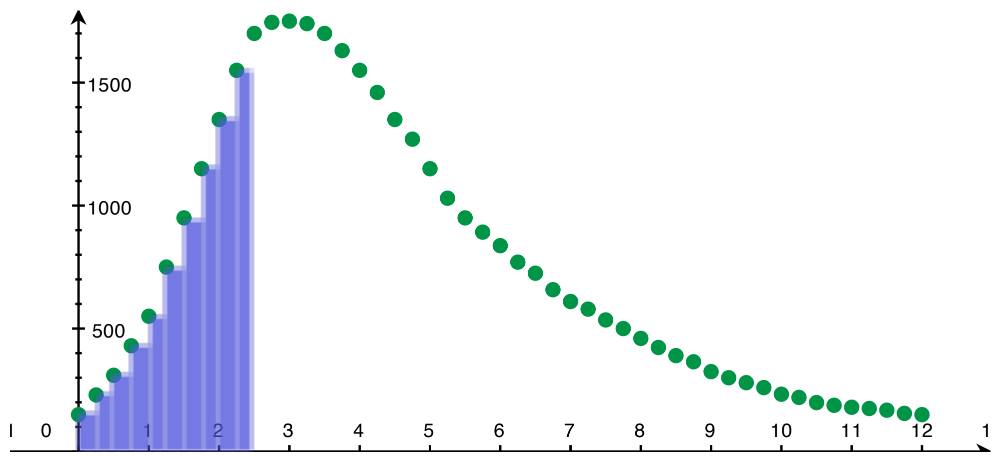
Example: estimating volume using data

A small dam breaks on a river. The average flow out of the stream is given by the following:

hours	m^3/s	hours	m^3/s	hours	m^3/s
0	150	4.25	1460	8.25	423
0.25	230	4.5	1350	8.5	390
0.5	310	4.75	1270	8.75	365
0.75	430	5	1150	9	325
1	550	5.25	1030	9.25	300
1.25	750	5.5	950	9.5	280
1.5	950	5.75	892	9.75	260
1.75	1150	6	837	10	233
2	1350	6.25	770	10.25	220
2.25	1550	6.5	725	10.5	199
2.5	1700	6.75	658	10.75	188
2.75	1745	7	610	11	180
3	1750	7.25	579	11.25	175
3.25	1740	7.5	535	11.5	168
3.5	1700	7.75	500	11.75	155
3.75	1630	8	460	12	150
4	1550				

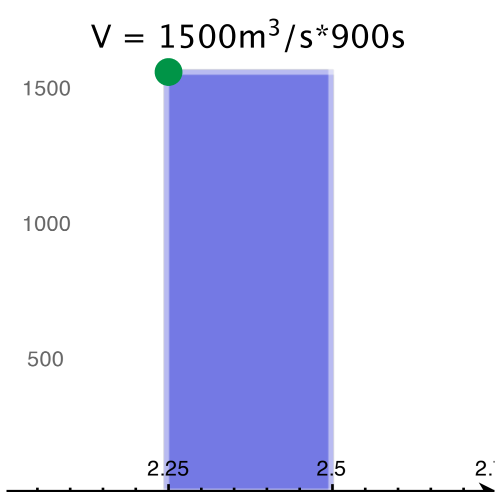
Over each time interval, we estimate the volume of water by

Average rate × 900 s



Over each time interval, we estimate the volume of water by

Average rate × 900 s



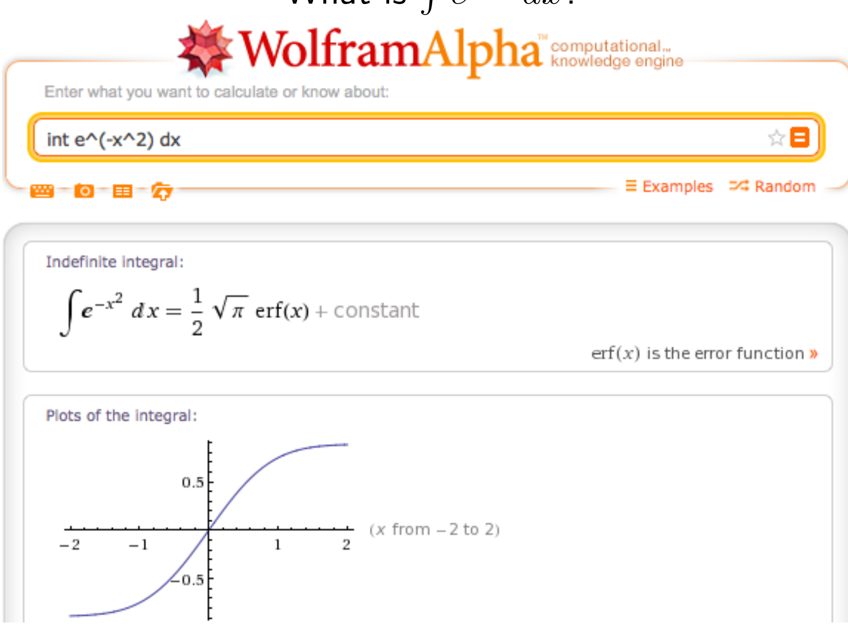
Over each time interval, we estimate the volume of water by

Average rate $\times$ 900 s

hours	m^3	hours	m^3	hours	m^3
0	135000	4.25	1314000	8.25	380700
0.25	207000	4.5	1215000	8.5	351000
0.5	279000	4.75	1143000	8.75	328500
0.75	387000	5	1035000	9	292500
1	495000	5.25	927000	9.25	270000
1.25	675000	5.5	855000	9.5	252000
1.5	855000	5.75	802800	9.75	234000
1.75	1035000	6	753300	10	209700
2	1215000	6.25	693000	10.25	198000
2.25	1395000	6.5	652500	10.5	179100
2.5	1530000	6.75	592200	10.75	169200
2.75	1570500	7	549000	11	162000
3	1575000	7.25	521100	11.25	157500
3.25	1566000	7.5	481500	11.5	151200
3.5	1530000	7.75	450000	11.75	139500
3.75	1467000	8	414000	12	135000
4	1395000			total=33,319,800	

Example: functions without nice antiderivatives

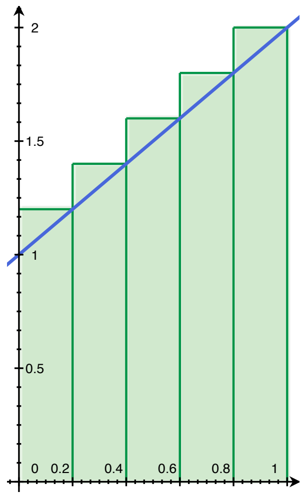
What is $\int e^{-x^2} dx$?



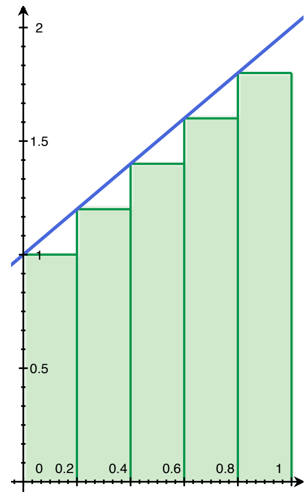
From Wikipedia: “In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape which occurs in probability, statistics and partial differential equations. ”

Upper and Lower Sums

Suppose we want to use rectangles to approximate the area under the graph of $y = x + 1$ on the interval $[0, 1]$.



Upper Riemann Sum



Lower Riemann Sum

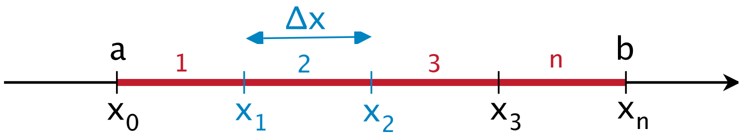
$$31/20 \quad > \quad 1.5 \quad > \quad 29/20$$

As you take more and more smaller and smaller rectangles, if f is nice, both of these will approach the real area.

n	U	L
100	1.505000000	1.495000000
150	1.503333333	1.496666667
200	1.502500000	1.497500000
300	1.501666667	1.498333333
500	1.501000000	1.499000000

In general: finding the Area Under a Curve

Let $y = f(x)$ be given and defined on an interval $[a, b]$.



Break the interval into n equal pieces.

Label the endpoints of those pieces $x_0, x_1, \ldots, x_n$.

Let $\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$ be the width of each interval.

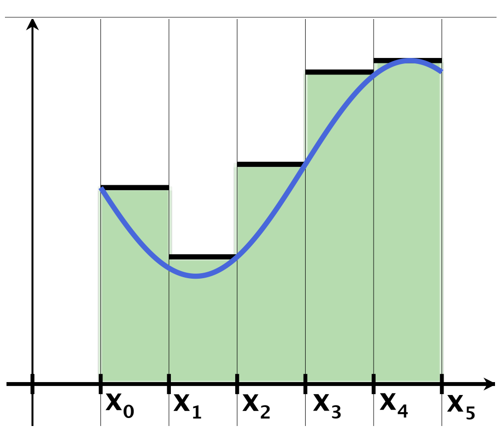
The **Upper Riemann Sum** is: let M_i be the *maximum* value of the function on that i^{th} interval, so

$$U(f, P) = M_1\Delta x + M_2\Delta x + \cdots + M_n\Delta x.$$

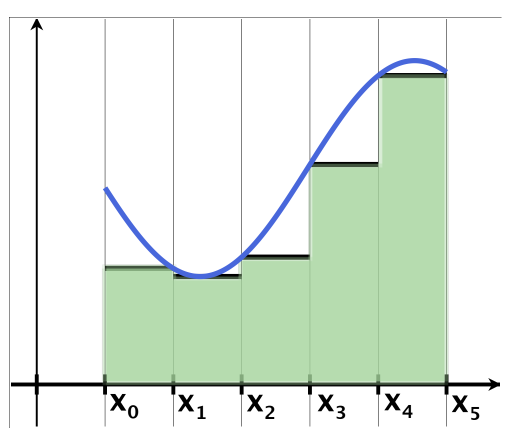
The **Lower Riemann Sum** is: let m_i be the *minimum* value of the function on that i^{th} interval, so

$$L(f, P) = m_1\Delta x + m_2\Delta x + \cdots + m_n\Delta x).$$

Take the limit as $n \to \infty$ or $\Delta x \to 0$.



Upper



Lower

Last time: sigma notation

If m and n are integers with $m \leq n$, and if f is a function defined on the integers from m to n, then the symbol $\sum_{i=m}^{n} f(i)$, called **sigma notation**, is means

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n)$$

Examples:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$\sum_{i=1}^{n} \sin(i) = \sin(1) + \sin(2) + \sin(3) + \cdots + \sin(n)$$

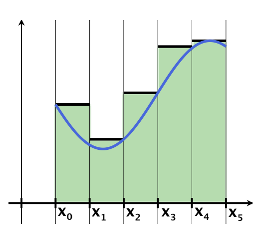
$$\sum_{i=0}^{n-1} x^i = x^0 + x + x^2 + x^2 + x^3 + x^4 + \cdots + x^{n-1}$$

The Area Problem Revisited

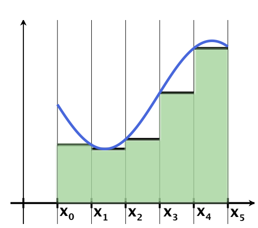
$$\text{Upper Riemann Sum} = \sum_{i=1}^{n} M_i \Delta x$$

$$\text{Lower Riemann Sum} = \sum_{i=1}^{n} m_i \Delta x,$$

where M_i and m_i are, respectively, the maximum and minimum values of f on the ith subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$.



Upper

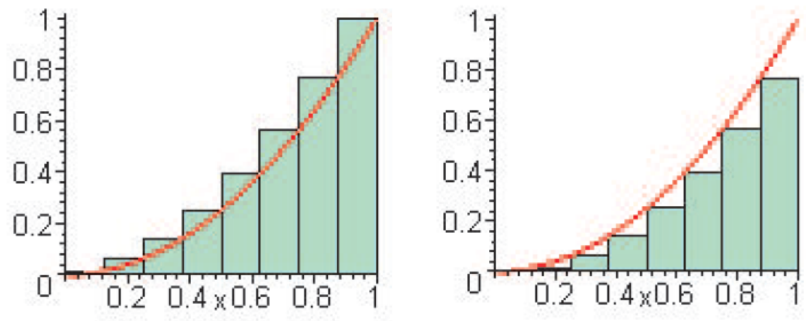


Lower

$\boxed{n = 5}$

Example

1. Write, in sigma notation, the upper and lower Riemann sums for the area under the graph of $f(x) = x^2$ on the interval $[0, 1]$, first with 8 subdivisions, and then with 10 subdivisions.



2. Write, in sigma notation, and estimate of the total displacement of a particle traveling along a straight line from $t = 1$ to $t = 5$, at a velocity of $v(t) = (t - 2)^3$, using 20 subdivisions.

The Definite Integral

We say that f is integrable on $[a, b]$ if there exists a number A such that

$$\text{Lower Riemann Sum} \leq A \leq \text{Upper Riemann Sum}$$

any number n of subdivisions. We write the number as

$$A = \int_a^b f(x)dx$$

and call it the **definite integral** of f over $[a, b]$.

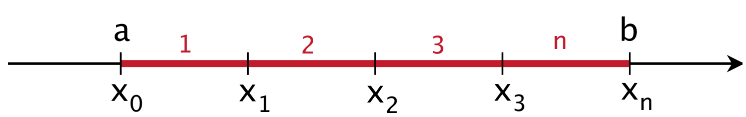
Trickiness: Who wants to find maxima/minima over every interval? Especially as $n \to \infty$? Calculus nightmare!!

More Riemann Sums

Let f be defined on $[a, b]$, and pick a positive integer n.
Let

$$\Delta x = \frac{b-a}{n}$$

Notice:



$$x + 0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \ldots$$

So let

$$\boxed{x_i = a + i * \Delta x.}$$

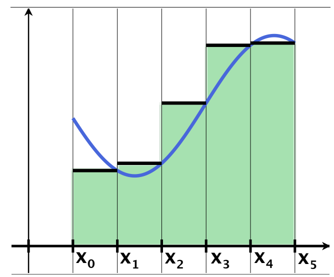
More Riemann Sums

Let f be defined on $[a,b]$, and pick a positive integer n.
Let

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i * \Delta x.$$

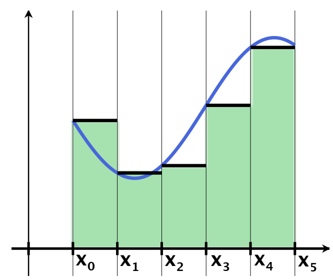
Then the Right Riemann Sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x,$$



and the Left Riemann Sum is

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i.$$



Integrals made easier

Theorem

If f is "Riemann integrable" on $[a,b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where c_i is **any** *point in the interval $[x_{i-1}, x_i]$.*

Punchline: We can calculate integrals by just using right or left sums! (instead of upper or lower sums)

Example: Set up left and right limit definitions of $\int_1^4 e^x dx$.
Remember that

> n is the number of pieces we've divided the interval into, and
>
> i indexes the terms in the sum (labels the rectangles).

Each piece:

$$\Delta x = \frac{4-1}{n} = \frac{3}{n} \qquad x_i = 1 + i * \Delta x = 1 + \frac{3i}{n}$$

So, the left Riemann sum is

$$\sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=0}^{n-1} e^{1+\frac{3i}{n}} \left(\tfrac{3}{n}\right) = \frac{3e}{n} \sum_{i=0}^{n-1} \left(e^{3/n}\right)^i$$

and the right Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} e^{1+\frac{3i}{n}} \left(\tfrac{3}{n}\right) = \frac{3e}{n} \sum_{i=1}^{n} \left(e^{3/n}\right)^i$$

So

$$\int_1^4 e^x dx = \lim_{n\to\infty} \frac{3e}{n} \sum_{i=0}^{n-1} \left(e^{3/n}\right)^i = \lim_{n\to\infty} \frac{3e}{n} \sum_{i=1}^{n} \left(e^{3/n}\right)^i.$$

On your own:

1. Set up the right limit definition of $\int_{-1}^5 \sin(x) dx$.

2. Rewrite the following expressions as $\int_a^b f(x) dx$ by identifying $f(x)$, a, and b. Also, identify if I've used the left or right Riemann sums.

   (a) $\lim_{n\to\infty} \sum_{i=0}^{n-1} \left((6 + \frac{7i}{n})^3 + 2\right) \left(\frac{7}{n}\right)$.

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2+\frac{i}{n}}{2-\frac{i}{n}} \left(\frac{1}{n}\right)$.

Recall from the reading that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Do we believe it?

$$n = 1: \quad \sum_{i=1}^{1} i = 1 = \frac{1(2)}{2} \qquad \checkmark$$

$$n = 2: \quad \sum_{i=1} i = 1 + 2 = 3 = \frac{2(3)}{2} \qquad \checkmark$$

$$n = 3: \quad \sum_{i=1}^{3} i = 1 + 2 + 3 = 6 = \frac{3(4)}{2} \qquad \checkmark$$

$$n = 4: \quad \sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10 = \frac{4(5)}{2} \qquad \checkmark$$

$$n = 5: \quad \sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15 = \frac{5(6)}{2} \qquad \checkmark$$

Recall from the reading that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Do we believe it?

$$n = 1: \quad \sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{1(2)(2+1)}{6} \qquad \checkmark$$

$$n = 2: \quad \sum_{i=1}^{2} i^2 = 1^2 + 2^2 = 5 = \frac{2(3)(4+1)}{6} \qquad \checkmark$$

$$n = 3: \quad \sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 14 = \frac{3(4)(6+1)}{6} \qquad \checkmark$$

$$n = 4: \quad \sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \frac{4(5)(8+1)}{6} \qquad \checkmark$$

$$n = 5: \quad \sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55 = \frac{5(6)(10+1)}{6} \qquad \checkmark$$

Recall from the reading that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Do we believe it?

$$n=1: \quad \sum_{i=1}^{1} i^3 = 1^3 = 1 = \left(\frac{1(2)}{2}\right)^2 \quad \checkmark$$

$$n=2: \quad \sum_{i=1}^{3} i^3 = 1^3 + 2^3 = 9 = \left(\frac{2(3)}{2}\right)^2 \quad \checkmark$$

$$n=3: \quad \sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 36 = \left(\frac{3(4)}{2}\right)^2 \quad \checkmark$$

$$n=4: \quad \sum_{i=1}^{4} i^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100 = \left(\frac{4(5)}{2}\right)^2 \quad \checkmark$$

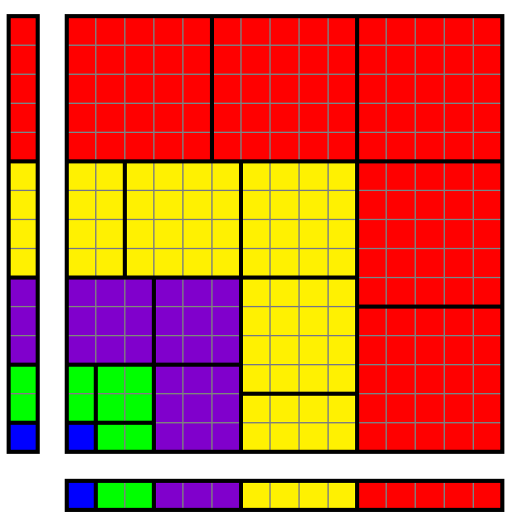
$$n=5: \quad \sum_{i=1}^{5} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = \left(\frac{5(6)}{2}\right)^2 \quad \checkmark$$

Recall from the reading that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Notice that this says

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 .$$



Now, let's compute

$$\int_1^3 5x^2\ dx.$$

Start by constructing the finite Riemann sum, with n subintervals:

Interval: $[1,3]$. $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

Endpoints: $x_i = a + \Delta x \cdot i = 1 + \frac{2}{n} \cdot i$

Rectangle area: $f(x_i)\Delta x = 5(1 + \frac{2}{n}i)^2 \left(\frac{2}{n}\right) = \frac{10}{n}(1 + 2 \cdot \frac{2i}{n} + \frac{2^2}{n^2}i^2)$

Finite Reimann sum: **Lots** of simplifying first!

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{10}{n}\left(1 + 2 \cdot \frac{2i}{n} + \frac{2^2}{n^2}i^2\right)$$

$$= \frac{10}{n}\sum_{i=1}^{n}\left(1 + \frac{4}{n}i + \frac{4}{n^2}i^2\right) = \frac{10}{n}\left(\sum_{i=1}^{n} 1 + \frac{4}{n}\sum_{i=1}^{n} i + \frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \frac{10}{n}\left(n + \frac{4}{n} \cdot \frac{n(n+1)}{2} + \frac{4}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right)$$

$$= 10 + 20\frac{n+1}{n} + \frac{20}{3} \cdot \frac{(n+1)(2n+1)}{n^2}$$

Now, let's compute

$$\int_1^3 5x^2\ dx.$$

Start by constructing the finite Riemann sum, with n subintervals:

$$\sum_{i=1}^{n} f(x_i)\Delta x = \cdots = \boxed{10 + 20\frac{n+1}{n} + \frac{20}{3} \cdot \frac{(n+1)(2n+1)}{n^2}}$$

Then, take the limit:

$$\int_1^3 5x^2\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\left(10 + 20\frac{n+1}{n} + \frac{20}{3} \cdot \frac{(n+1)(2n+1)}{n^2}\right)$$

$$= \boxed{10 + 20 \cdot 1 + \frac{20}{3} \cdot 2}.$$

You try: Compute

$$\int_3^7 2x^2 - x\ dx.$$