Antiderivatives and Initial Value Problems

Definition: An antiderivative of a function f on an interval $[a, b]$ is another function F such that $F'(x) = f(x)$ for all x in $[a, b]$.

Examples:

1. An antiderivative of $f(x) = 2x$ is $F(x) = x^2$.
2. Another antiderivative of $f(x) = 2x$ is $F(x) = x^2 + 1$.
3. There are *lots* of antiderivatives of $f(x) = 2x$ which look like $F(x) = x^2 + C$.

Finding all antiderivatives

Suppose that h is differentiable and $h'(x) = 0$ for all x . Then h is a constant function, i.e. $h(x) = C$, for some fixed number C.

Now, suppose $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$, i.e. $F'(x) = f(x)$ and $G'(x) = f(x)$. Then $F(x)$ and $G(x)$ are both differentiable (their derivatives are $f(x)$), so $G(x) - F(x)$ is differentiable. Also,

$$\frac{d}{dx}(G(x) - F(x)) = G'(x) - F'(x) = f(x) - f(x) = 0.$$

So $G(x) - F(x) = C$ for some fixed number C, i.e.

$$G(x) = F(x) + C.$$

In summary: If $F(x)$ is one antiderivative of $f(x)$, then every other antiderivative must be of the form $F(x) + C$ for some number C.

Example: **All** of the antiderivatives of $f(x) = 2x$ look like

$$F(x) = x^2 + C$$

for some constant C.

Notice: Every function $f(x)$ that has at least one antiderivative $F(x)$ has **infinitely many** antiderivatives

$$F(x) + C.$$

(We say $\frac{dy}{dx} = f(x)$ has an *infinite family of solutions* (for y).)

We refer to $F(x) + C$ as the general antiderivative or the indefinite integral of $f(x)$, and denote it by

$$F(x) + C = \int f(x)\ dx.$$

Example:

$$\int 2x\ dx = x^2 + C.$$

Examples

$$\int x^2 \; dx = \tfrac{1}{3}x^3 + C, \qquad \text{because} \qquad \tfrac{d}{dx}(\tfrac{1}{3}x^3 + C) = \tfrac{1}{3} * 3x^2 = x^2$$

$$\int x^3 \; dx = \tfrac{1}{4}x^4 + C, \qquad \text{because} \qquad \tfrac{d}{dx}(\tfrac{1}{4}x^4 + C) = \tfrac{1}{4} * 4x^3 = x^3$$

$$\int x^5 \; dx =$$

$$\int x^{-3} \; dx =$$

$$\int x^k \; dx =$$

Some important basic integrals

$$\int x^k \; dx = \frac{1}{k+1}x^{k+1} + C \qquad \text{if } k \neq -1 \; *$$

$$\int \sin(x) \; dx =$$

$$\int \cos(x) \; dx =$$

$$\int e^x \; dx =$$

$$\int \sec^2(x) \; dx =$$

$$\int \frac{1}{\sqrt{1-x^2}} \; dx =$$

*The antiderivative of $1/x$

Note: $1/x$ is defined over all real numbers $\neq 0$, but $\ln(x)$ is only defined over positive real numbers!

Calculate $\frac{d}{dx} \ln|x|$.

Recall

$$|x| = \begin{cases} x & x \geq 0, \\ -x & x < 0. \end{cases}$$

So (1) the domain of $\ln|x|$ is $(-\infty, 0) \cup (0, \infty)$, and

$$(2) \quad \ln|x| = \begin{cases} \ln(x) & x \geq 0, \\ \ln(-x) & x < 0. \end{cases}$$

Compute the derivative one piece at a time, using chain rule:

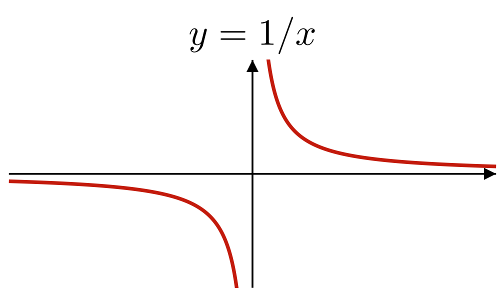
$$\frac{d}{dx} \ln|x| = \begin{cases} 1/x & x \geq 0, \\ -(1/(-x)) = 1/x & x < 0, \end{cases} = 1/x.$$

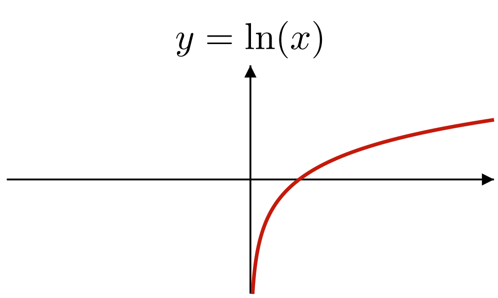
Therefore,

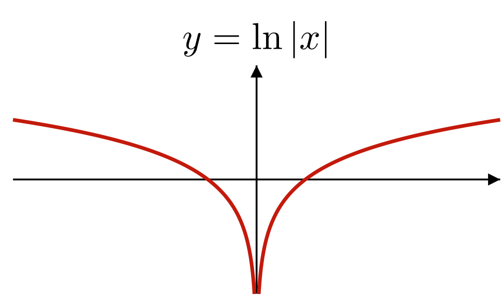
$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

*The antiderivative of $1/x$

Note: $1/x$ is defined over all real numbers $\neq 0$, but $\ln(x)$ is only defined over positive real numbers!







Theorem (Opposite of sum and constant rules). Suppose the functions $f(x)$ and $g(x)$ both have antiderivatives on the interval $[a, b]$. Then for any constant a, the functions $af(x)$ and $f(x) + g(x)$ have antiderivatives on $[a, b]$, given by

$$\int af(x)\ dx = a \int f(x)\ dx$$

and

$$\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$$

Example: Since $\int 2x\ dx = x^2 + A$ and $\int \cos(x)\ dx = \sin(x) + B$, we have

$$\int 2x + 3\cos(x)\ dx = \int 2x\ dx + \int 3\cos(x)\ dx$$

$$= \int 2x\ dx + 3 \int \cos(x)\ dx = x^2 + 3\sin(x) + C.$$

To come later: "Opposite of chain rule" is called u-substitution, and "opposite of product rule" is called integration by parts. For now: most derivative rules make a mess; antiderivatives have to clean up that mess, which is not always possible using elementary functions!

Differential equations

A **differential equation** is an equation involving derivatives.
The *goal* is usually to solve for y.
Just like you could use algebra to solve

$$y^2 + x^2 = 1$$

for y, you can use calculus (and algebra) to solve things like

$$\frac{dy}{dx} - 5y = 0 \qquad \text{for } y.$$

A solution to a differential equation is a function you can plug in that satisfies the equation.
For example, $y = e^{5x}$ is a solution to the differential equation above since

$$\frac{d}{dx}e^{5x} = 5e^{5x},$$

so

$$\frac{dy}{dx} - 5y = (5e^{5x}) - 5(e^{5x}) = 0. \quad \checkmark$$

Simplest differential equations: antiderivatives

Finding an antiderivative can also be thought of as solving a differential equation:

"Solve the differential equation $\frac{d}{dx}y = x^2$."

Answer: $\quad y = \int x^2 dx = \frac{1}{3}x^3 + C.$

Check: $\quad \frac{d}{dx}\frac{1}{3}x^3 + C. = \frac{1}{3} * 3 * x^2 + 0 = x^2 \quad \checkmark$

Examples

(1) Solve the differential equation $\quad y' = 2x + \sin(x).$

(2) Check that $\cos(x) + \sin(x)$ is a solution to $\quad \dfrac{d^2y}{dx^2} + y = 0.$

Definition

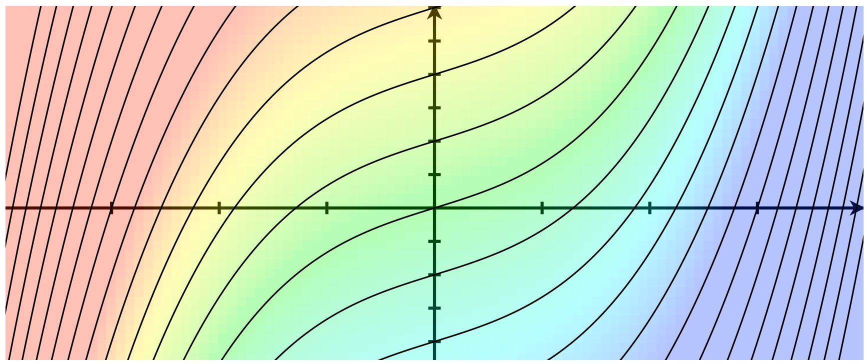
An initial-value problem is a differential equation together with enough additional conditions to specify the constants of integration that appear in the general solution.

The particular solution of the problem is then a function that satisfies both the differential equation and also the additional conditions.

Initial value problems

Find a solution to the differential equation $\frac{d}{dx}y = x^2 + 1$ which *also* satisfies $y(2) = 8/3$.

$$\boxed{\text{general solution:} \quad y = \frac{1}{3}x^3 + x + C}$$



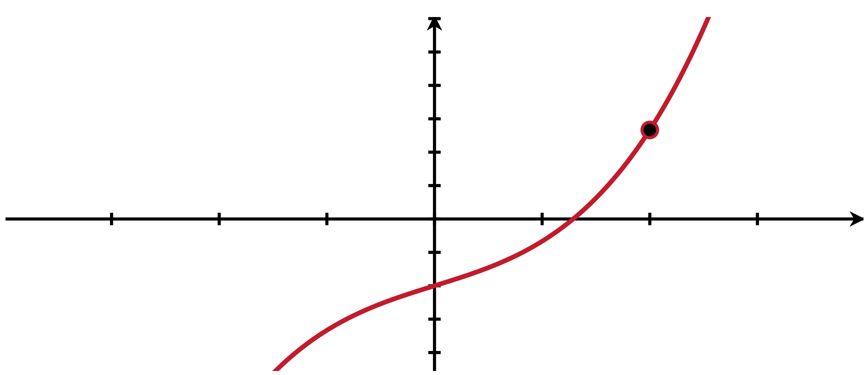
Each color corresponds to a choice of C.

Initial value problems

Find a solution to the differential equation $\frac{d}{dx}y = x^2 + 1$ which *also* satisfies $y(2) = 8/3$.

$$\boxed{\text{general solution:} \quad y = \frac{1}{3}x^3 + x + C}$$

$$\boxed{\text{particular solution:} \quad y = \frac{1}{3}x^3 + x - 2}$$



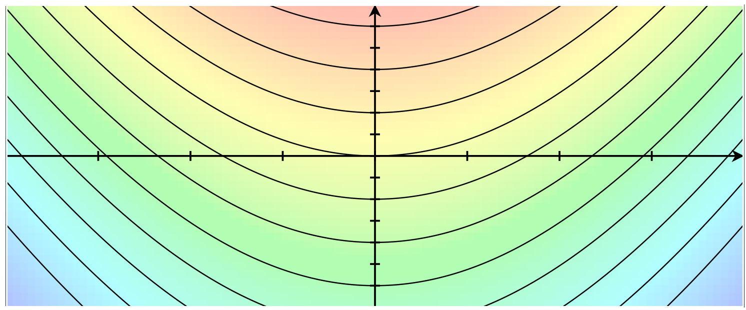
Each color corresponds to a choice of C.
Red cuve is the *particular* solution.

Solve the initial value problem

$$\frac{dy}{dx} = 2x + \sin(x)$$

subject to $y(0) = 0$.

$$\boxed{\text{general solution:} \quad y = x^2 - \cos(x) + C}$$

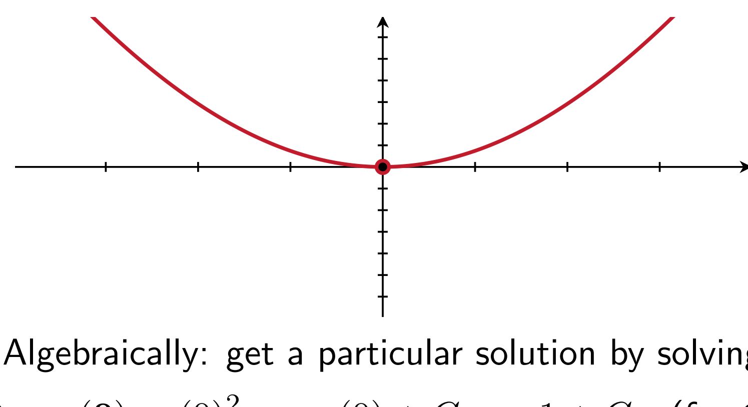


Solve the initial value problem

$$\frac{dy}{dx} = 2x + \sin(x)$$

subject to $y(0) = 0$.

$$\boxed{\text{general solution:} \quad y = x^2 - \cos(x) + C}$$



Algebraically: get a particular solution by solving

$$\mathbf{0} = \mathbf{y}(\mathbf{0}) = (0)^2 - \cos(0) + C = -1 + C \quad \text{(for } C\text{)}$$

$$C = 1, \qquad \text{so } \boxed{y = x^2 - \cos(x) + 1.}$$

Solve the initial-value problem $y'' = \cos x,\ y'(\frac{\pi}{2}) = 2,\ y(\frac{\pi}{2}) = 3\pi.$

Step 1: Calculate the antiderivative of $\cos(x)$ to find the general solution for y'.

Step 2: Plug in the values $y'(\frac{\pi}{2}) = 2$ to calculate C.

Step 3: Write down the *particular* solution for y'.

Step 4: Calculate the antiderivative of your particular solution in Step 3 to find the *general solution for* y.

Step 5: Plug in the values $y(\frac{\pi}{2}) = 3\pi$ to solve for the new constant.

Step 6: Write down the *particular* solution for y.

Word problem:

An object dropped from a cliff has acceleration $a = -9.8\ m/sec^2$ under the influence of gravity. What is the function $s(t)$ that models its height at time t?

Initial value problem:

Solve

$$\frac{d^2s}{dt^2} = -9.8, \qquad s(0) = s_0,\ s'(0) = 0.$$

Word problem:

Suppose that a baseball is thrown upward from the roof of a 100 meter high building. It hits the street below eight seconds later. What was the initial velocity of the baseball, and how high did it rise above the street before beginning its descent?

Initial value problem:

Solve

$$\frac{d^2s}{dt^2} = -9.8, \qquad s(0) = 100,\ s(8) = 0.$$

Use your solution to

(1) calculate $s'(0)$, and

(2) solve $s'(t_1) = 0$ for t_1 and calculate $s(t_1)$.

And now for something completely different...

(Help with reading sections 5.1 and 5.2.)

$$1+2=3=\tfrac{2}{2}\cdot(2+1)$$
$$1+2+3=6=\tfrac{3}{2}\cdot(3+1)$$
$$1+2+3+4=10=\tfrac{4}{2}\cdot(4+1)$$
$$1+2+3+4+5=15=\tfrac{5}{2}\cdot(5+1)$$
$$\begin{aligned}1+2+\cdots+99+100 &= (1+100)+(2+99)+(3+98)\\ &\quad+\cdots+(50+51)\\ &= 101+101+101+\cdots+101\\ &= 50\cdot 101=\boxed{\frac{100}{2}\cdot(100+1)}\end{aligned}$$
$$\begin{aligned}1+2+\cdots+100+101 &= (1+101)+(2+100)+(3+99)\\ &\quad+\cdots+(50+52)+51\\ &= 102+102+102+\cdots+102+51\\ &= 50\cdot 102+51=(50.5)\cdot 102=\boxed{\frac{101}{2}\cdot(101+1)}\end{aligned}$$

And now for something completely different...

(Help with reading sections 5.1 and 5.2.)

$$\begin{aligned}1+2+\cdots+99+100 &= (1+100)+(2+99)+(3+98)+\cdots+(50+51)\\ &= 101+101+101+\cdots+101\\ &= 50\cdot 101=\boxed{\frac{100}{2}\cdot(100+1)}\end{aligned}$$
$$\begin{aligned}1+2+\cdots+100+101 &= (1+101)+(2+100)+(3+99)+\cdots+(50+52)+51\\ &= 102+102+102+\cdots+102+51\\ &= 50\cdot 102+51=(50.5)\cdot 102=\boxed{\frac{101}{2}\cdot(101+1)}\end{aligned}$$

Thing 1: In general, for $n \geq 0$ an integer,
$$1+2+\cdots+n=\frac{n}{2}(n+1).$$

Thing 2: We have a notation for writing long sums compactly, called sigma notation.

Sigma notation

Let $f(x)$ be a function of integers, and let $a \leq b$ be integers. Then

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(b).$$

Namely,

$\sum$	means add up (sum)
$i =$	means i is your variable (index)
$i = a$	means start plugging in at a
$\sum^{b}$	means stop plugging in at b
$f(i)$	means plug i into $f(x)$

Sigma notation

Let $f(x)$ be a function of integers, and let $a \leq b$ be integers. Then

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(b).$$

For example,

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5,$$

$$\sum_{i=-2}^{5} i^2 = (-2)^1 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2,$$

and

$$\sum_{i=0}^{3} i^2 + 1 = (0^2 + 1) + (1^2 + 1) + (2^2 + 1) + (3^2 + 1)$$

$$= (0^2 + 1^2 + 2^2 + 3^2) + (1 + 1 + 1 + 1) = \sum_{i=0}^{3} i^2 + \sum_{i=0}^{3} 1.$$

Sigma notation: some identities

We've seen

$$\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$$

$$\sum_{i=a}^{b} f(i) + g(i) = \sum_{i=a}^{b} f(i) + \sum_{i=a}^{b} g(i).$$

Distributive law:

$$\sum_{i=a}^{b} cf(i) = c\sum_{i=a}^{b} f(i).$$

More identities in Section 5.2.

Why??? It turns out that antiderivatives are related to area trapped between curves. We'll both see why this is true, and learn how to estimate antiderivatives when computing them exactly is impossible. (Think: derivative rules are great, but sometimes you just *need* limits.)