You try:

To summarize, we have 7 indeterminate form types:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad \text{and} \quad 1^\infty.$$

For each of the following limits, decide if the limit is an indeterminate form. If so, identify which indeterminate form it is.

1. $\lim_{x \to \infty} x - \ln(x)$
2. $\lim_{x \to 0^+} x - \ln(x)$
3. $\lim_{x \to \infty} x^x$
4. $\lim_{x \to 0^+} x^x$
5. $\lim_{x \to \infty} (1/x)^x$
6. $\lim_{x \to 0^+} (1 + \sin(x))^{\cot(x)}$
7. $\lim_{x \to \pi/2^+} \sec(x) - \tan(x)$

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x \to a} G(x) = L$, then

$$\lim_{x \to a} F(G(x)) = F\left(\lim_{x \to a} G(x)\right) = F(L).$$

In particular, since $F(x) = \ln(x)$ is continuous,

$$\ln\left(\lim_{x \to a} G(x)\right) = \lim_{x \to a} \ln(G(x)).$$

Since $\ln(x)$ is invertible over the positive real line, if I can compute the limit of $\ln(G(x))$, then I can solve for the limit of $G(x)$.

Why do I like this? Logarithms turn exponentials into products!

$$\ln(f(x)^{g(x)}) = g(x)\ln(f(x))$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x \to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x \to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x \to 0^+} x^x\right) = \lim_{x \to 0^+} \ln(x^x) = \lim_{x \to 0^+} x\ln(x).$$

Now we've changed this into the indeterminate form $0 \cdot \infty$, which we know how to solve! We saw before that

$$\lim_{x \to 0^+} x\ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} -x = 0.$$

So

$$\ln(L) = 0, \quad \text{implying } L = e^0 = 1.$$

So $\boxed{\lim_{x \to 0^+} x^x = 1}$.

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a} g(x)\ln(f(x)).$$

Step 2: Simplify, and if necessary, use L'Hospital's rule to calculate $\lim_{x\to a} g(x)\ln(f(x)) = M$.

Step 3: Finally, $\ln(L) = M$ implies $L = e^M$ solves for L.

You try: Calculate the following limits.

$$(1)\ \lim_{x\to\infty} x^{e^{-x}}, \quad (2)\ \lim_{x\to\infty} (e^x)^{1/\ln(x)}$$

$$(3)\ \lim_{x\to 0^+} (1+\sin(x))^{\cot(x)} \quad (4)\ \lim_{x\to 0^+} (1+\sin(3x))^{\cot(x)}$$

(recall, $\cos(0) = 1$, $\sin(0) = 0$)

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty} (e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x\to\infty} e^{x/\ln(x)} = \exp\left(\lim_{x\to\infty} x/\ln(x)\right)$$

$$\overset{\text{L'H}}{=\!=} \exp\left(\lim_{x\to\infty} 1/(1/x)\right) = \exp\left(\lim_{x\to\infty} x\right) = \infty.$$

Moral: There are no exact rules for how to do these problems. There are just lots of strategies. Get lots of practice!

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $L = \lim_{x\to 0^+} \csc(x) - \cot(x)$. Note
   $$\csc(x) - \cot(x) = \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)} = \frac{1-\cos(x)}{\sin(x)}.$$
   So by L'Hospital,
   $$L = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} \stackrel{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{\sin(x)}{\cos(x)} = 0.$$

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1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note
   $$x - \sqrt{x^2-x} = (x - \sqrt{x^2-x})\left(\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}}\right)$$
   $$= \frac{(x)^2 - (\sqrt{x^2-x})^2}{x+\sqrt{x^2-x}} = \frac{x^2 - x^2 + x}{x+\sqrt{x^2-x}} = \frac{x}{x+\sqrt{x^2-x}}.$$
   So
   $$L = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{1}{1+\sqrt{1-1/x}} = 1/2$$

Solving indeterminate forms of type $\infty - \infty$

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   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2 - x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$. Start with
   $$e^L = \exp\left(\lim_{x\to\infty} \ln(x) - x^2\right) = \lim_{x\to\infty} e^{\ln(x)-x^2}.$$
   Then since $e^{\ln(x)-x^2} = e^{\ln(x)}/e^{x^2} = x/e^{x^2}$, we have
   $$e^L = \lim_{x\to\infty} x/e^{x^2} = 0, \qquad \text{so } L = -\infty.$$

You try:

For each, find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

1. $\lim\limits_{x\to 0^+} \sin^{-1}(x)/x$
2. $\lim\limits_{x\to 1} \dfrac{x}{x-1} - \dfrac{1}{\ln(x)}$
3. $\lim\limits_{x\to\infty} x\sin(\pi/x)$
4. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$
5. $\lim\limits_{x\to\infty} x^{\ln(2)/(1+\ln(x))}$
6. $\lim\limits_{x\to 0} \dfrac{\tan(x)}{\tanh(x)}$

Note: For extra practice, go back and prove the claims on the slides with the graphical examples.

Switching gears: Optimization Warm up

Sketch the graph of

$$f(x) = (x-3)(x-2)(x-1) = x^3 - 6x^2 + 11x - 6$$

over the interval $[1, 4]$. Mark any critical points and inflection points. What is the absolute maximum over this interval? What is the absolute minimum over this interval?
[useful value: $\sqrt{3}/3 \approx .6$]

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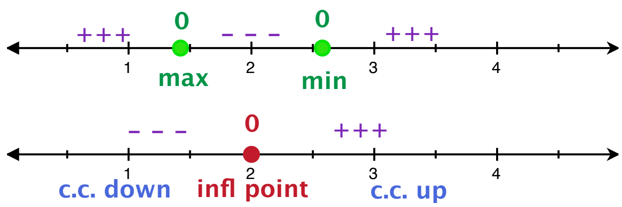
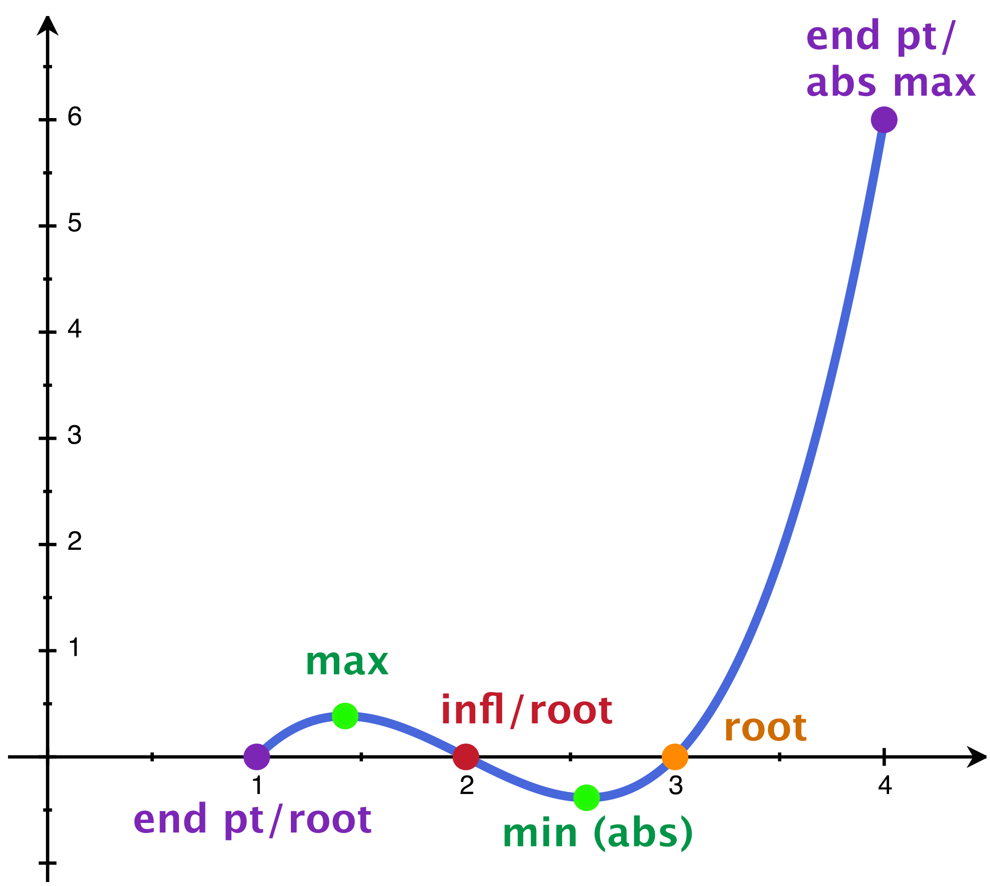
$$f'(x) = 3x^2 - 12x + 11 = 3\left(x - (2 - \sqrt{3}/3)\right)\left(x - (2 + \sqrt{3}/3)\right)$$

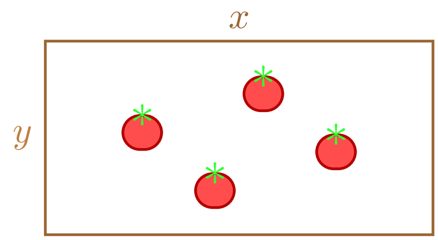
$$f''(x) = 6x - 12 = 6(x-2)$$



$f(2 - \sqrt{3}/3) \approx 0.385$
$f(2 + \sqrt{3}/3) \approx -0.385$
$f(1) = 0$
$f(4) = 6$



Suppose you want to fence off a garden, and you have 100m of fence. What is the largest area that you can fence off?



Get it into math:

Know: $2x + 2y = 100$ Want: Maximize $A = xy$

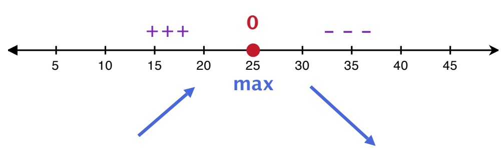
Problem: The area, xy, is a function of two variables!!
Strategy: Use the first equation to get xy into one variable:
Solve $2x + 2y = 100$ (the "constraint") and plug into xy (the function you want to optimize).

$$2x + 2y = 100 \implies y = 50 - x$$

so $xy = x(50 - x) = 50x - x^2$. $\boxed{\text{Domain: } 0 \le x \le 50}$

$\boxed{\text{New problem: Maximize } A(x) = 50x - x^2 \text{ over the interval } 0 < x < 50.}$

Solution: $A'(x) = 50 - 2x$
So the only critical point is when $50 - 2x = 0$, so $\boxed{x = 25}$.
Three strategies for finding the maximum:
(1) First derivative test:



$\boxed{\text{So } A(25) = 625 \text{ is maximal.}}$

(2) Pretend we're on a closed interval, then throw out the endpoints:

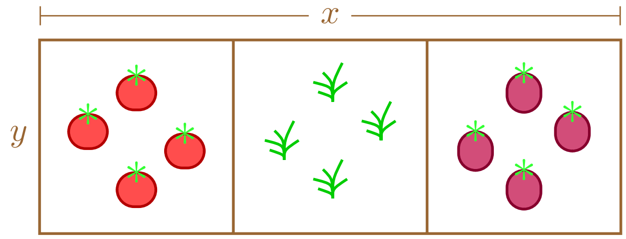
x	$A(x)$	
25	$\boxed{625}$	max!
0	0	min
50	0	min

Since the maximum is not at one of the points I have to throw out, it must me a maximum on the open interval (there is no absolute minimum over the open interval $(0, 50)$).

(3) Second derivative test:

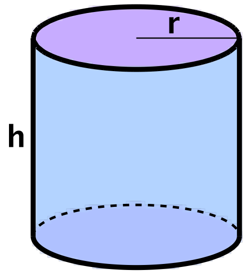
$A''(x) = -2 < 0$ so $A(25) = 625$ must be a maximum.

Now suppose, instead, you want to divide your plot up into three equal parts:



If you still only have 100 m of fence, what is the largest area that you can fence off?

Suppose you want to make a can which holds about 16 ounces (28.875 in^3). If the material for the top and bottom of the can costs 4 ¢/in^2 and the material for the sides of the can costs 3 ¢/in^2. What is the minimum cost for the can?



Put into math:
Constraint: $V = \pi r^2 h = 28.875$.
Cost: $4 *$ (SA of top + bottom) $+3 *$ (SA of side)

Top: πr^2 Bottom: πr^2 Sides: $(2\pi r)h$

$$\boxed{\text{Total cost: } C = 4 * 2 * (\pi r^2) + 3 * ((2\pi r)h)}$$

Get into one variable: Use the constraint!

$$\pi r^2 h = 28.875 \implies h = \frac{28.875}{\pi} r^{-2} \implies C(r) = 8\pi r^2 + 6\pi r \left(\frac{28.875}{\pi} r^{-2}\right)$$

So

$$\boxed{C(r) = 8\pi r^2 + 6 * 28.875 r^{-1}}$$

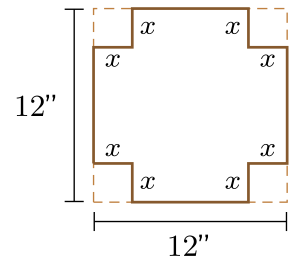
(Domain: $r > 0$)

$$\boxed{\text{New problem: Minimize } C(r) = 8\pi r^2 + 6 * 28.875 r^{-1} \text{ for } r > 0.}$$

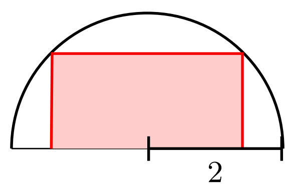
[hint: If you don't have a calculator, use the second derivative test!]

You try

1. An open-top box is to be made by cutting small congruent squares from the corners of a 12" $\times$12" sheet of cardboard and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



2. A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



Hint: