You try:

To summarize, we have 7 indeterminate form types:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad \text{and} \quad 1^\infty.$$

For each of the following limits, decide if the limit is an indeterminate form. If so, identify which indeterminate form it is.

1. $\lim_{x\to\infty} x - \ln(x)$
2. $\lim_{x\to 0^+} x - \ln(x)$
3. $\lim_{x\to\infty} x^x$
4. $\lim_{x\to 0^+} x^x$
5. $\lim_{x\to\infty} (1/x)^x$
6. $\lim_{x\to 0^+} (1+\sin(x))^{\cot(x)}$
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1. $\lim_{x \to \infty} x - \ln(x)$ Ans: type $\infty - \infty$
2. $\lim_{x \to 0^+} x - \ln(x) = 0 - (-\infty) = \infty$ Ans: not indet
3. $\lim_{x \to \infty} x^x = \infty$ Ans: not indet
4. $\lim_{x \to 0^+} x^x$ Ans: type 0^0
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7. $\lim_{x \to \pi/2^+} \sec(x) - \tan(x)$ Ans: type $\infty - \infty$

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x\to a} G(x) = L$, then

$$\lim_{x\to a} F(G(x)) = F\left(\lim_{x\to a} G(x)\right) = F(L).$$

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Why do I like this? Logarithms turn exponentials into products!

$$\ln(f(x)^{g(x)}) = g(x)\ln(f(x))$$

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(1) $\lim_{x\to\infty} x^{e^{-x}}$, (2) $\lim_{x\to\infty} (e^x)^{1/\ln(x)}$

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Answers: 1, ∞, e, e^3.

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

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Moral: There are no exact rules for how to do these problems. There are just lots of strategies. Get lots of practice!

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   So by L'Hospital,

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$$L = \lim_{x\to\infty} \frac{x}{x + \sqrt{x^2 - x}}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{1}{1 + \sqrt{1 - 1/x}}$$

Solving indeterminate forms of type $\infty - \infty$

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You try:

For each, find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

1. $\lim_{x \to 0^+} \sin^{-1}(x)/x$
2. $\lim_{x \to 1} \frac{x}{x-1} - \frac{1}{\ln(x)}$
3. $\lim_{x \to \infty} x \sin(\pi/x)$
4. $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$
5. $\lim_{x \to \infty} x^{\ln(2)/(1+\ln(x))}$
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Note: For extra practice, go back and prove the claims on the slides with the graphical examples.

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3. $\lim_{x\to\infty} x\sin(\pi/x)$ — Ans: π
4. $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$ — Ans: 3
5. $\lim_{x\to\infty} x^{\ln(2)/(1+\ln(x))}$ — Ans: 2
6. $\lim_{x\to 0} \frac{\tan(x)}{\tanh(x)}$ — Ans: 1

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Switching gears: Optimization Warm up

Sketch the graph of

$$f(x) = (x-3)(x-2)(x-1) = x^3 - 6x^2 + 11x - 6$$

over the interval $[1, 4]$. Mark any critical points and inflection points. What is the absolute maximum over this interval? What is the absolute minimum over this interval?
[useful value: $\sqrt{3}/3 \approx .6$]

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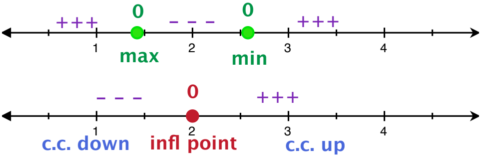
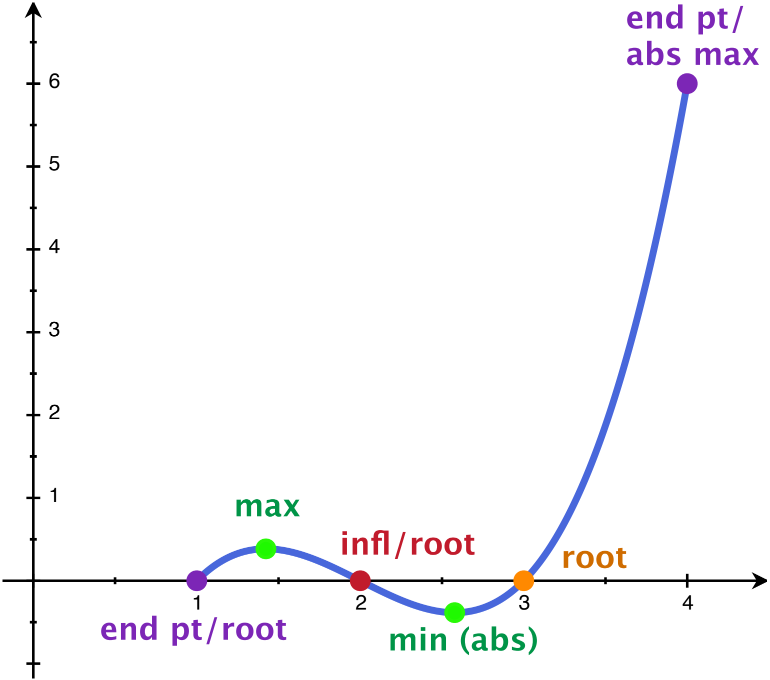
$$f'(x) = 3x^2 - 12x + 11 = 3\left(x - (2 - \sqrt{3}/3)\right)\left(x - (2 + \sqrt{3}/3)\right)$$

$$f''(x) = 6x - 12 = 6(x-2)$$

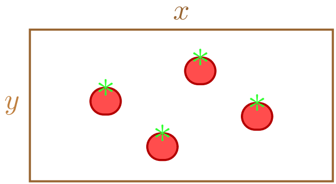


$f(2 - \sqrt{3}/3) \approx 0.385$
$f(2 + \sqrt{3}/3) \approx -0.385$
$f(1) = 0$
$f(4) = 6$



end pt/
abs max
6
5
4
3
2
1
max
infl/root
root
1
2
3
4
end pt/root
min (abs)

Suppose you want to fence off a garden, and you have 100m of fence. What is the largest area that you can fence off?



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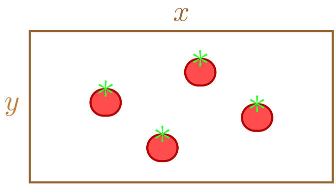

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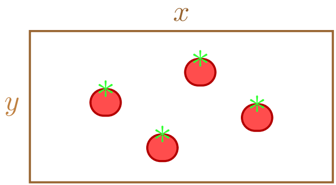

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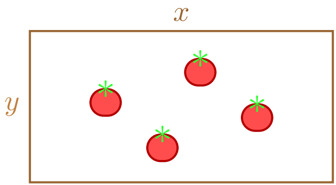

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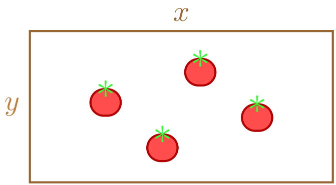

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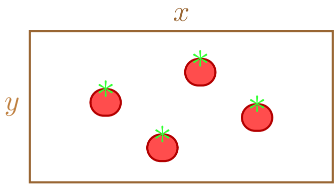

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So the only critical point is when $50 - 2x = 0$, so $\boxed{x = 25}$.

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(1) First derivative test:

(2) Pretend we're on a closed interval, then throw out the endpoints:

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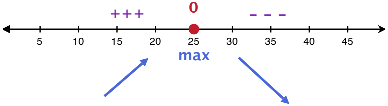

So $A(25) = 625$ is maximal.

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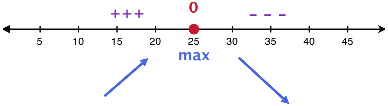

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(2) Pretend we're on a closed interval, then throw out the endpoints:

x	$A(x)$	
25	625	max!
0	0	min
50	0	min

Since the maximum is not at one of the points I have to throw out, it must me a maximum on the open interval (there is no absolute minimum over the open interval $(0, 50)$).

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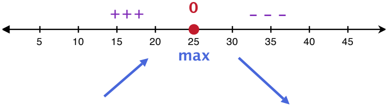

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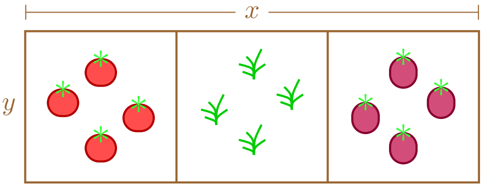
$A''(x) = -2 < 0$ so $A(25) = 625$ must be a maximum.

Now suppose, instead, you want to divide your plot up into three equal parts:



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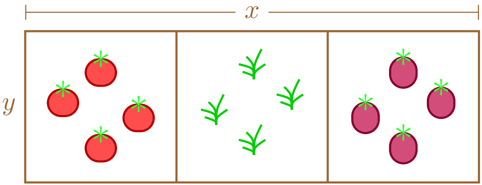


If you still only have 100 m of fence, what is the largest area that you can fence off?

Solution:

Constraint: $2x + 4y = 100$, so $y = 25 - \frac{1}{2}x$.

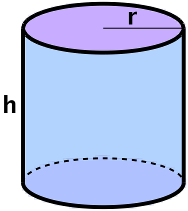
Maximize: $A = xy$ over $0 < x < 50$.

Plug in constraint: $A(x) = x(25 - x/2) = 25x - x^2/2$

Find critical points: $0 = A'(x) = 25 - x$, so $x = 25$.

Second derivative test: $A''(x) = -1 < 0$ so $\boxed{A(25) = 25 * 12.5}$ is a maximum.

Suppose you want to make a can which holds about 16 ounces (28.875 in^3). If the material for the top and bottom of the can costs 4 ¢/in^2 and the material for the sides of the can costs 3 ¢/in^2. What is the minimum cost for the can?



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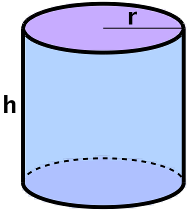

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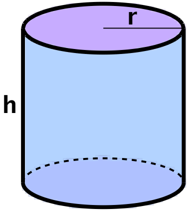

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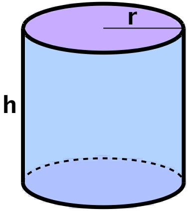

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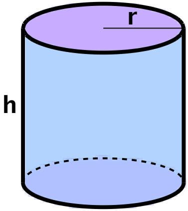

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$$\pi r^2 h = 28.875 \implies h = \frac{28.875}{\pi} r^{-2}$$

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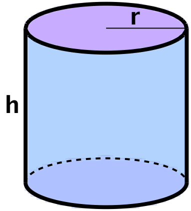

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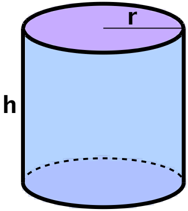
Get into one variable: Use the constraint!

$$\pi r^2 h = 28.875 \implies h = \frac{28.875}{\pi} r^{-2} \implies C(r) = 8\pi r^2 + 6\pi r \left(\frac{28.875}{\pi} r^{-2} \right)$$

So

$$C(r) = 8\pi r^2 + 6 * 28.875 r^{-1}$$

Suppose you want to make a can which holds about 16 ounces (28.875 in^3). If the material for the top and bottom of the can costs 4 ¢/in^2 and the material for the sides of the can costs 3 ¢/in^2. What is the minimum cost for the can?



Put into math:

Constraint: $V = \pi r^2 h = 28.875.$

Cost: $4 *$ (SA of top + bottom) $+3 *$ (SA of side)

Top: πr^2 Bottom: πr^2 Sides: $(2\pi r)h$

Total cost: $C = 4 * 2 * (\pi r^2) + 3 * ((2\pi r)h)$

Get into one variable: Use the constraint!

$$\pi r^2 h = 28.875 \implies h = \frac{28.875}{\pi} r^{-2} \implies C(r) = 8\pi r^2 + 6\pi r \left(\frac{28.875}{\pi} r^{-2} \right)$$

So

$$C(r) = 8\pi r^2 + 6 * 28.875 r^{-1}$$

(Domain: $r > 0$)

New problem: Minimize $C(r) = 8\pi r^2 + 6 * 28.875 r^{-1}$ for $r > 0$.

[hint: If you don't have a calculator, use the second derivative test!]

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[hint: If you don't have a calculator, use the second derivative test!]

Solution: $(6 * 28.875 = 173.25)$

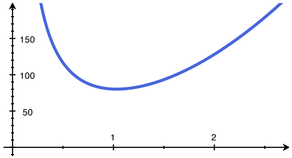
$$C'(r) = 16\pi r - 173.25 r^{-2} = \frac{1}{r^2}\left(16\pi r^3 - 173.25\right)$$

Critical point: $C'(r) = \sqrt[3]{\frac{173.25}{16\pi}} \approx 1.031$

Second derivative test: $C''(r) = 16\pi + 173.25 r^{-3} > 0$ when $r > 0$,

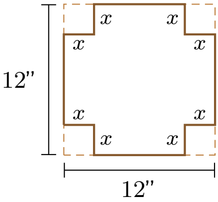
so $C(r)$ is concave up, and so $C\left(\sqrt[3]{\frac{173.25}{16\pi}}\right)$ is a minimum.

Minimal value: $C\left(\sqrt[3]{\frac{173.25}{16\pi}}\right) \approx 80.2041$¢

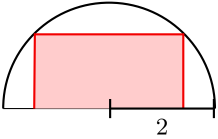


You try

1. An open-top box is to be made by cutting small congruent squares from the corners of a 12" $\times$12" sheet of cardboard and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



2. A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



Hint: