Warm up

Compute the following limits.

1. $\lim_{x \to 0} \frac{\sin(2x)}{x}$
2. $\lim_{x \to 0} \frac{\cos(x) - 1}{x^2}$
3. $\lim_{x \to \infty} \frac{3x^2 + 2x - 1}{5x^2 - 7}$
4. $\lim_{x \to \infty} \frac{3x^2 + 2x - 1}{5x - 7}$
5. $\lim_{x \to 0} \frac{x}{x + 1}$

More on limits, indeterminate forms, and L'Hospital's rule

Consider the function

$$F(x) = \frac{\ln(x)}{x-1}.$$

As $x \to 1$, both the numerator and the denominator approach 0. Both approach somewhat slowly, but does one go faster than the other? Or does it approach some interesting ratio? Similar question for $x \to \infty$, where both the numerator and denominator approach ∞.

Indeterminate forms are ratios where the numerator and the denominator each either approach 0, or each approach $\pm\infty$.
So far, we've been able to calculate limits with indeterminate forms through algebraic tricks or substitution, or recognizing limits as derivatives.

Past examples of solving indeterminate forms

1. $\displaystyle\lim_{x\to\infty} \frac{3x^2+x}{5x^2-1}\left(\frac{x^{-2}}{x^{-2}}\right) = \lim_{x\to\infty} \frac{3+x^{-1}}{5-x^{-2}} = \frac{3}{5}$
2. $\displaystyle\lim_{x\to-\infty} \frac{3e^{2x}+e^{x}}{5e^{2x}-e^{x}}\left(\frac{e^{-x}}{e^{-x}}\right) = \lim_{x\to\infty} \frac{3e^{x}+1}{5e^{x}-1} = \frac{0+1}{0-1} = -1$
3. $\displaystyle\lim_{x\to\pi} \frac{e^{\sin(x)}-1}{x-\pi}$

   Recall, $f'(a) = \displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

   Note, $e^{\sin(x)}\Big|_{x=\pi} = e^{\sin(\pi)} = e^0 = 1$.

   So

$$\lim_{x\to\pi} \frac{e^{\sin(x)}-1}{x-\pi} = \frac{d}{dx}e^{\sin(x)}\Big|_{x=\pi} = \cos(x)e^{\sin(x)}\Big|_{x=\pi} = (-1)e^0 = -1.$$

So similarly, since $\ln(1) = 0$,

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \frac{d}{dx}\ln(x)\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1.$$

L'Hospital's rule

L'Hospital's rule relates the limit of the ratio of two functions to the limit of the ratio of their derivatives.
Consider differentiable functions $f(x)$ and $g(x)$ such that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x),$$

and $g'(x) \neq 0$ for x close to but not equal to a. Then

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{f(x)}{x - a}, \text{ and}$$

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a} = \lim_{x\to a} \frac{g(x)}{x - a}.$$

(If f or g are not defined at a, we can still work around this. . .)
So

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f(x)(x-a)}{(x-a)g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}.$$

L'Hospital's rule

Theorem
Suppose f and g are differentiable functions and $g'(x) \neq 0$ for x close to but not equal to a. Suppose that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \quad \text{or} \quad \lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x).$$

Then if the limit of $f'(x)/g'(x)$ as $x \to a$ exists (or is $\pm\infty$), we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The same holds for $x \to \pm\infty$ and one-sided limits $x \to a^{\pm}$.

Example. Let's recheck $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.
$\ln(x)$ and $x - 1$ differentiable? ✓ $g'(x) = 1 \neq 0$ ✓,
$\ln(x) \to 0$ and $x - 1 \to 0$ as $x \to 1$ ✓

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1 \checkmark$$

L'Hospital's rule: if f and g are differentiable, $g'(x) \neq 0$ near a (but $g'(a) = 0$ is ok), and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x).$$

Same goes for one-sided limits and $x \to \pm\infty$.

You try: For each of the following, verify that you can use L'Hospital's rule to calculate the limit, and then do so.

$$(1)\ \lim_{x\to\pi} \frac{e^{\sin(x)} - 1}{x - \pi} \qquad (2)\ \lim_{x\to\infty} \frac{e^x}{x} \qquad (3)\ \lim_{x\to\infty} \frac{\ln(x)}{x}$$

Each of the following has some reason why you can't use L'Hospital's rule. For each, what is the reason?

$$(1)\ \lim_{x\to 0} \frac{x}{|x|} \qquad (2)\ \lim_{x\to 0^+} \frac{x}{\lfloor x \rfloor} \qquad (3)\ \lim_{x\to\pi} \frac{\sin(x)}{1 - \cos(x)}$$

(Recall, $\lfloor x \rfloor$ is the *floor* function, and gives back the biggest integer less than or equal to x, i.e. $\lfloor 2.1 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$, $\lfloor 1 \rfloor = 1$, etc..)

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{\frac{3}{2}x^{1/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{2}{3}\frac{e^x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{4}{3}e^x x^{1/2} = \infty$$

For any a, there is some n for which $\frac{d^n}{dx^n}x^a$ is some constant times x^{a-n} such that $a - n \leq 0$. So

$$\lim_{x\to\infty} \frac{e^x}{x^a} = \infty \quad \text{for all } a!$$

You try: For what a does $x^a / \ln(x)$ approach ∞ as $x \to \infty$?

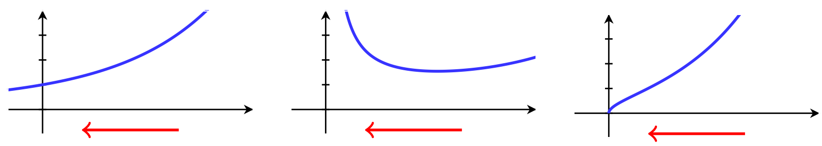
Other indeterminate forms

Our first two indeterminate forms were

(1) f/g if $f, g \to \pm\infty$ and (2) f/g if $f, g \to 0$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen. For example, as $x \to 0^+$,

$\frac{e^x - 1}{x} \to 1$ $\frac{e^x - 1}{x^2} \to \infty$ $\frac{e^x - 1}{\sqrt{x}} \to 0$



To this list, we add

(3) fg if $f \to 0$ and $g \to \pm\infty$

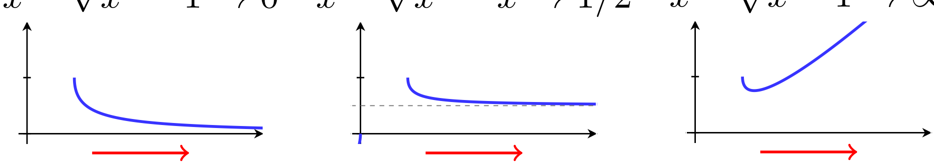
Notice, if $g(x) \to 0^{\pm}$, then $1/g(x) \to \pm\infty$.
Example: Compute $\lim_{x \to 0^+} x \ln(x)$. We rewrite this as

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = -\lim_{x \to 0^+} x = 0.$$

Other indeterminate forms

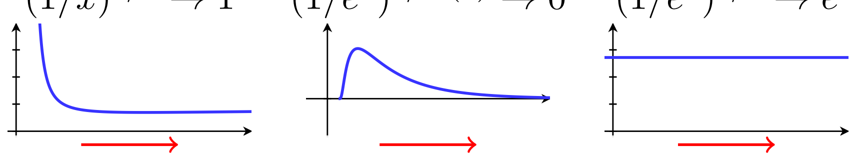
(4) $f - g$ if $f, g \to \infty$ (called type $\infty - \infty$)
For example, as $x \to \infty$,

$x - \sqrt{x^2 - 1} \to 0$ $x - \sqrt{x^2 - x} \to 1/2$ $x - \sqrt{x - 1} \to \infty$



(5) f^g if $f, g \to 0$ (called type 0^0)
For example, as $x \to \infty$,

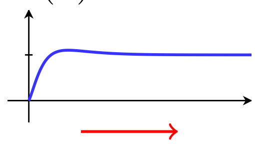
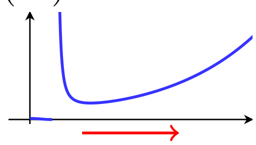
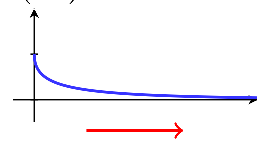
$(1/x)^{1/x} \to 1$ $(1/e^x)^{1/\ln(x)} \to 0$ $(1/e^x)^{1/x} \to e$



Other indeterminate forms

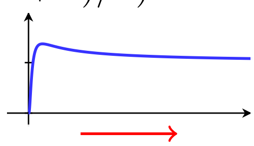
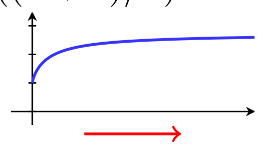
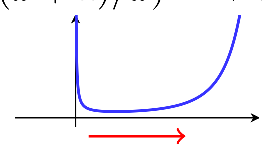
(6) f^g if $f \to \infty$ and $g \to 0$ (called type ∞^0)
For example, as $x \to \infty$,

$(x)^{e^{-x}} \to 1$ $\quad$ $(e^x)^{1/\ln(x)} \to \infty$ $\quad$ $(e^x)^{-1/\sqrt{x}} \to 0$

(7) f^g if $f \to 1$ and $g \to \infty$ (called type 1^∞)
For example, as $x \to \infty$,

$((x+1)/x)^{\ln(x)} \to 1$ $\quad$ $((x+1)/x)^x \to e$ $\quad$ $((x+1)/x)^{e^x} \to \infty$

You try:

To summarize, we have 7 indeterminate form types:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad \text{and} \quad 1^\infty.$$

For each of the following limits, decide if the limit is an indeterminate form. If so, identify which indeterminate form it is. I

1. $\lim_{x \to \infty} x - \ln(x)$
2. $\lim_{x \to 0^+} x - \ln(x)$
3. $\lim_{x \to \infty} x^x$
4. $\lim_{x \to 0^+} x^x$
5. $\lim_{x \to \infty} (1/x)^x$
6. $\lim_{x \to 0^+} (1 + \sin(x))^{\cot(x)}$
7. $\lim_{x \to \pi/2^+} \sec(x) - \tan(x)$

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x\to a} G(x) = L$, then

$$\lim_{x\to a} F(G(x)) = f\left(\lim_{x\to a} G(x)\right) = F(L).$$

In particular, since $F(x) = \ln(x)$ is continuous,

$$\ln\left(\lim_{x\to a} G(x)\right) = \lim_{x\to a} \ln(G(x)).$$

Since $\ln(x)$ is invertible over the positive real line, if I can compute the limit of $\ln(G(x))$, then I can solve for the limit of $G(x)$.

Why do I like this? Logarithms turn exponentials into products!

$$\ln(f(x)^{g(x)}) = g(x)\ln(f(x))$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x\to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x).$$

Now we've changed this into the indeterminate form $0\cdot\infty$, which we know how to solve! We saw before that

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

So

$$\ln(L) = 0, \quad \text{implying } L = e^0 = 1.$$

So $\boxed{\lim_{x\to 0^+} x^x = 1}$.

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a} g(x)\ln(f(x)).$$

Step 2: Simplify, and if necessary, use L'Hospital's rule to calculate $\lim_{x\to a} g(x)\ln(f(x)) = M$.

Step 3: Finally, $\ln(L) = M$ implies $L = e^M$ solves for L.

You try: Calculate the following limits.

$$(1)\ \lim_{x\to\infty} x^{e^{-x}}, \quad (2)\ \lim_{x\to\infty} (e^x)^{1/\ln(x)}$$

$$(3)\ \lim_{x\to 0^+} (1+\sin(x))^{\cot(x)} \quad (4)\ \lim_{x\to 0^+} (1+\sin(3x))^{\cot(x)}$$

(recall, $\cos(0) = 1$, $\sin(0) = 0$)

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty}(e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x\to\infty} e^{x/\ln(x)} = \exp\left(\lim_{x\to\infty} x/\ln(x)\right)$$

$$\overset{\text{L'H}}{=\!=} \exp\left(\lim_{x\to\infty} 1/(1/x)\right) = \exp\left(\lim_{x\to\infty} x\right) = \infty.$$

Moral: There are no exact rules for how to do these problems. There are just lots of strategies. Get lots of practice!

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $L = \lim_{x\to 0^+} \csc(x) - \cot(x)$. Note

$$\csc(x) - \cot(x) = \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)} = \frac{1-\cos(x)}{\sin(x)}.$$

   So by L'Hospital,

$$L = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{\sin(x)}{\cos(x)} = 0.$$

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   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2-x} = (x - \sqrt{x^2-x})\left(\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2-x})^2}{x+\sqrt{x^2-x}} = \frac{x^2-x^2+x}{x+\sqrt{x^2-x}} = \frac{x}{x+\sqrt{x^2-x}}.$$

   So

$$L = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{1}{1+\sqrt{1-1/x}} = 1/2$$

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   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2-x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$. Start with

$$e^L = \exp\left(\lim_{x\to\infty} \ln(x) - x^2\right) = \lim_{x\to\infty} e^{\ln(x)-x^2}.$$

   Then since $e^{\ln(x)-x^2} = e^{\ln(x)}/e^{x^2} = x/e^{x^2}$, we have

$$e^L = \lim_{x\to\infty} x/e^{x^2} = 0, \qquad \text{so } L = -\infty.$$

You try:

For each, find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

1. $\lim_{x \to 0^+} \sin^{-1}(x)/x$
2. $\lim_{x \to 1} \dfrac{x}{x-1} - \dfrac{1}{\ln(x)}$
3. $\lim_{x \to \infty} x \sin(\pi/x)$
4. $\lim_{x \to 0} \dfrac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$
5. $\lim_{x \to \infty} x^{\ln(2)/(1+\ln(x))}$
6. $\lim_{x \to 0} \dfrac{\tan(x)}{\tanh(x)}$

Note: For extra practice, go back and prove the claims on the slides with the graphical examples.