Warm up

Compute the following limits.

1. $\lim_{x\to 0} \frac{\sin(2x)}{x}$
2. $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$
3. $\lim_{x\to \infty} \frac{3x^2+2x-1}{5x^2-7}$
4. $\lim_{x\to \infty} \frac{3x^2+2x-1}{5x-7}$
5. $\lim_{x\to 0} \frac{x}{x+1}$

More on limits, indeterminate forms, and L'Hospital's rule

Consider the function

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As $x \to 1$, both the numerator and the denominator approach 0. Both approach somewhat slowly, but does one go faster than the other? Or does it approach some interesting ratio? Similar question for $x \to \infty$, where both the numerator and denominator approach ∞.

Indeterminate forms are ratios where the numerator and the denominator each either approach 0, or each approach $\pm\infty$. So far, we've been able to calculate limits with indeterminate forms through algebraic tricks or substitution, or recognizing limits as derivatives.

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But what about

$$\lim_{x\to\infty} \frac{\ln(x)}{x-1}??$$

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L'Hospital's rule relates the limit of the ratio of two functions to the limit of the ratio of their derivatives.
Consider differentiable functions $f(x)$ and $g(x)$ such that

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(If f or g are not defined at a, we can still work around this. . .)

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Suppose f and g are differentiable functions and $g'(x) \neq 0$ for x close to but not equal to a. Suppose that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \quad \textit{or} \quad \lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x).$$

Then if the limit of $f'(x)/g'(x)$ as $x \to a$ exists (or is $\pm\infty$), we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The same holds for $x \to \pm\infty$ and one-sided limits $x \to a^{\pm}$.

Example. Let's recheck $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.
$\ln(x)$ and $x-1$ differentiable? ✓ $g'(x) = 1 \neq 0$ ✓,
$\ln(x) \to 0$ and $x-1 \to 0$ as $x \to 1$ ✓

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1}$$

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$\ln(x)$ and $x-1$ differentiable? ✓ $g'(x) = 1 \neq 0$ ✓,
$\ln(x) \to 0$ and $x - 1 \to 0$ as $x \to 1$ ✓

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1 \checkmark$$

L'Hospital's rule: if f and g are differentiable, $g'(x) \neq 0$ near a (but $g'(a) = 0$ is ok), and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \text{ or } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty,$$

then

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x).$$

Same goes for one-sided limits and $x \to \pm\infty$.

You try: For each of the following, verify that you can use L'Hospital's rule to calculate the limit, and then do so.

(1) $\displaystyle \lim_{x \to \pi} \frac{e^{\sin(x)} - 1}{x - \pi}$ (2) $\displaystyle \lim_{x \to \infty} \frac{e^x}{x}$ (3) $\displaystyle \lim_{x \to \infty} \frac{\ln(x)}{x}$

Each of the following has some reason why you can't use L'Hospital's rule. For each, what is the reason?

(1) $\displaystyle \lim_{x \to 0} \frac{x}{|x|}$ (2) $\displaystyle \lim_{x \to 0^+} \frac{x}{\lfloor x \rfloor}$ (3) $\displaystyle \lim_{x \to \pi} \frac{\sin(x)}{1 - \cos(x)}$

(Recall, $\lfloor x \rfloor$ is the *floor* function, and gives back the biggest integer less than or equal to x, i.e. $\lfloor 2.1 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$, $\lfloor 1 \rfloor = 1$, etc..)

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x}$$

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

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$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

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$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{\frac{3}{2}x^{1/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{2}{3} \frac{e^x}{\frac{1}{2}x^{-1/2}}$$

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For any a, there is some n for which $\frac{d^n}{dx^n}x^a$ is some constant times x^{a-n} such that $a - n \leq 0$. So

$$\lim_{x\to\infty} \frac{e^x}{x^a} = \infty \quad \text{for all } a!$$

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You try: For what a does $x^a/\ln(x)$ approach ∞ as $x \to \infty$?

Other indeterminate forms

Our first two indeterminate forms were

$$\text{(1) } f/g \quad \text{if} \quad f, g \to \pm\infty \quad \text{and} \quad \text{(2) } f/g \quad \text{if} \quad f, g \to 0$$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen.

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(1) f/g if $f, g \to \pm\infty$ and (2) f/g if $f, g \to 0$

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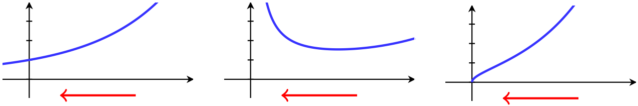
$$\frac{e^x-1}{x} \to 1 \qquad \frac{e^x-1}{x^2} \to \infty \qquad \frac{e^x-1}{\sqrt{x}} \to 0$$



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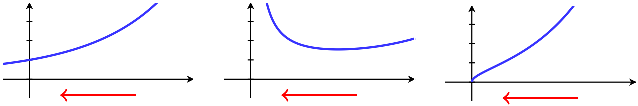

To this list, we add

$$(3)\ fg \text{ if} \quad f \to 0 \text{ and } g \to \pm\infty$$

Notice, if $g(x) \to 0^{\pm}$, then $1/g(x) \to \pm\infty$.

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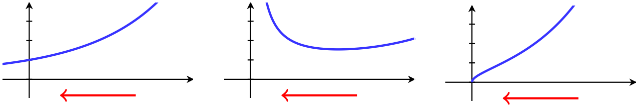

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Notice, if $g(x)\to 0^{\pm}$, then $1/g(x)\to\pm\infty$.
Example: Compute $\lim_{x\to 0^+} x\ln(x)$.

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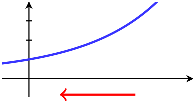
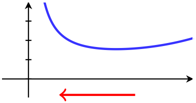
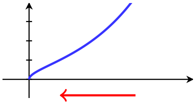

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Notice, if $g(x) \to 0^{\pm}$, then $1/g(x) \to \pm\infty$.
Example: Compute $\lim_{x\to 0^+} x\ln(x)$. We rewrite this as

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}}$$

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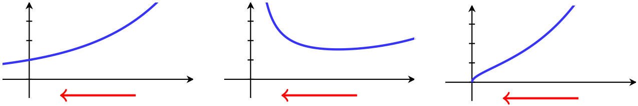

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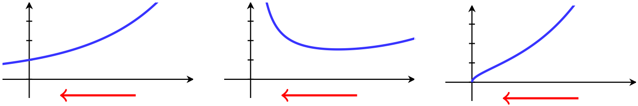

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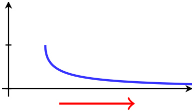
Example: Compute $\lim_{x \to 0^+} x \ln(x)$. We rewrite this as

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = -\lim_{x \to 0^+} x = 0.$$

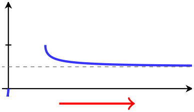
Other indeterminate forms

(4) $f - g$ if $f, g \to \infty$ (called type $\infty - \infty$)
For example, as $x \to \infty$,

$x - \sqrt{x^2 - 1} \to 0$



$x - \sqrt{x^2 - x} \to 1/2$



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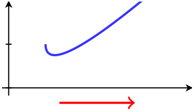

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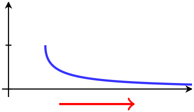
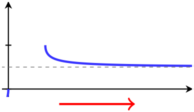
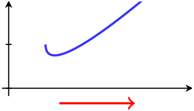


$x - \sqrt{x^2 - x} \to 1/2$

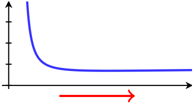


$x - \sqrt{x - 1} \to \infty$

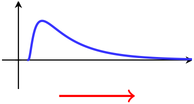


(5) f^g if $f, g \to 0$ (called type 0^0)
For example, as $x \to \infty$,

$(1/x)^{1/x} \to 1$



$(1/e^x)^{1/\ln(x)} \to 0$



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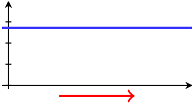
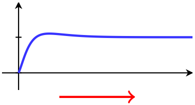


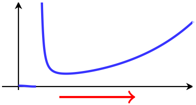
Other indeterminate forms

(6) f^g if $f \to \infty$ and $g \to 0$ (called type ∞^0)
For example, as $x \to \infty$,

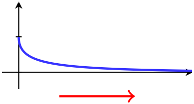
$$(x)^{e^{-x}} \to 1$$



$$(e^x)^{1/\ln(x)} \to \infty$$



$$(e^x)^{-1/\sqrt{x}} \to 0$$



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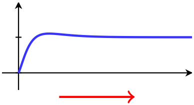

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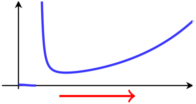

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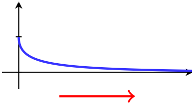
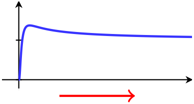


(7) f^g if $f \to 1$ and $g \to \infty$ (called type 1^∞)
For example, as $x \to \infty$,

$((x+1)/x)^{\ln(x)} \to 1$



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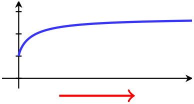

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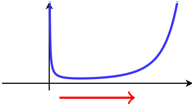

You try:

To summarize, we have 7 indeterminate form types:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad \text{and} \quad 1^\infty.$$

For each of the following limits, decide if the limit is an indeterminate form. If so, identify which indeterminate form it is. I

1. $\lim_{x \to \infty} x - \ln(x)$
2. $\lim_{x \to 0^+} x - \ln(x)$
3. $\lim_{x \to \infty} x^x$
4. $\lim_{x \to 0^+} x^x$
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7. $\lim_{x \to \pi/2^+} \sec(x) - \tan(x)$

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1. $\lim_{x \to \infty} x - \ln(x)$ Ans: type $\infty - \infty$
2. $\lim_{x \to 0^+} x - \ln(x) = 0 - (-\infty) = \infty$ Ans: not indet
3. $\lim_{x \to \infty} x^x = \infty$ Ans: not indet
4. $\lim_{x \to 0^+} x^x$ Ans: type 0^0
5. $\lim_{x \to \infty} (1/x)^x = 0$ Ans: not indet! (see 5.8#52)
6. $\lim_{x \to 0^+} (1 + \sin(x))^{\cot(x)}$ Ans: type 1^∞
7. $\lim_{x \to \pi/2^+} \sec(x) - \tan(x)$ Ans: type $\infty - \infty$

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Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x\to a} G(x) = L$, then

$$\lim_{x\to a} F(G(x)) = f\left(\lim_{x\to a} G(x)\right) = F(L).$$

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Why do I like this? Logarithms turn exponentials into products!

$$\ln(f(x)^{g(x)}) = g(x)\ln(f(x))$$

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Answers: 1, ∞, e, e^3.

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Moral: There are no exact rules for how to do these problems. There are just lots of strategies. Get lots of practice!

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   So by L'Hospital,
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   $$e^L = \lim_{x\to\infty} x/e^{x^2} = 0, \qquad \text{so } L = -\infty.$$

You try:

For each, find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

1. $\lim_{x \to 0^+} \sin^{-1}(x)/x$
2. $\lim_{x \to 1} \frac{x}{x-1} - \frac{1}{\ln(x)}$
3. $\lim_{x \to \infty} x \sin(\pi/x)$
4. $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$
5. $\lim_{x \to \infty} x^{\ln(2)/(1+\ln(x))}$
6. $\lim_{x \to 0} \frac{\tan(x)}{\tanh(x)}$

Note: For extra practice, go back and prove the claims on the slides with the graphical examples.

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4. $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$ Ans: 3
5. $\lim_{x \to \infty} x^{\ln(2)/(1+\ln(x))}$ Ans: 2
6. $\lim_{x \to 0} \frac{\tan(x)}{\tanh(x)}$ Ans: 1

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