L'Hospital's rule: if f and g are differentiable, $g'(x) \neq 0$ near a (but $g'(a) = 0$ is ok), and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x).$$

Same goes for one-sided limits and $x \to \pm\infty$.

You try: Use whatever methods you have at your disposal to calculate the following limits.

$$(1)\ \lim_{x\to 0^+} \frac{\sin(x)}{1-\cos(x)}, \quad (2)\ \lim_{x\to 0} \frac{3^x - e^x}{x}, \quad (3)\ \lim_{x\to 0} \frac{2x^2+x}{3x^2+1},$$

$$(4)\ \lim_{x\to\infty} x^{-\ln(x)}, \quad (5)\ \lim_{x\to\infty} \tan^{-1}(e^x/x), \quad (6)\ \lim_{x\to-\infty} \tan^{-1}(e^x/x),$$

$$(7)\ \lim_{x\to\infty} e^{-x}\tan^{-1}(x), \quad (8)\ \lim_{x\to 0^-} \frac{x}{|x|}, \quad (9)\ \lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2}.$$

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Answers: ∞, $\ln(3) - 1$, 0, 0, $\pi/2$, 0, 0, -1, 0.

Answers

1. $\lim_{x \to 0^+} \frac{\sin(x)}{1 - \cos(x)}$
2. $\lim_{x \to 0} \frac{3^x - e^x}{x}$
3. $\lim_{x \to 0} \frac{2x^2 + x}{3x^2 + 1}$
4. $\lim_{x \to \infty} x^{-\ln(x)}$
5. $\lim_{x \to \infty} \tan^{-1}(e^x/x)$

Answers

1. $\lim_{x\to 0^+} \frac{\sin(x)}{1-\cos(x)} \stackrel{\text{L'H}}{=\!=} \lim_{x\to 0} \frac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x\to 0} \frac{3^x - e^x}{x}$
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$$\lim_{x\to\infty} u = \lim_{x\to\infty} e^x/x \overset{\text{L'H}}{=\!=} e^x/1 = \infty.$$

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1. $\lim_{x\to 0^+} \frac{\sin(x)}{1-\cos(x)} \overset{\text{L'H}}{=\!=} \lim_{x\to 0} \frac{\cos(x)}{\sin(x)} = \infty$
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Thus

$$\lim_{x\to\infty} \tan^{-1}(e^x/x) = \lim_{u\to\infty} \tan^{-1}(u) = \pi/2.$$

Answers

6. $\lim_{x \to -\infty} \tan^{-1}(e^x/x)$

7. $\lim_{x \to \infty} e^{-x} \tan^{-1}(x)$

8. $\lim_{x \to 0^-} \frac{x}{|x|}$.

9. $\lim_{x \to \infty} \frac{\ln(\sqrt{x})}{x^2}$

Answers

6. $\lim_{x\to -\infty} \tan^{-1}(e^x/x)$: As $x \to -\infty$, $e^x/x \to 0$, so

$$\lim_{x\to -\infty} \tan^{-1}(e^x/x) = \tan^{-1}(0) = 0.$$

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7. $\lim_{x \to \infty} e^{-x} \tan^{-1}(x) = 0$, since $\tan^{-1}(x)$ stays between $\pm\pi/2$ and $e^{-x} \to 0$ as $x \to \infty$.
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