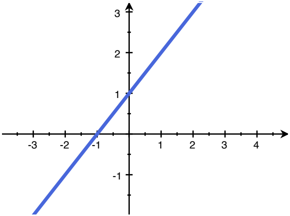
Curve Sketching

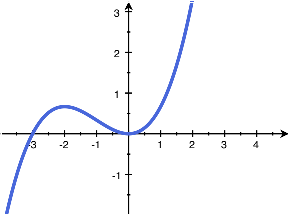
Warm up

Below are pictured six functions: f, f', f'', g, g', and g''. Pick out the two functions that could be f and g, and match them to their first and second derivatives, respectively.

(a)



(b)



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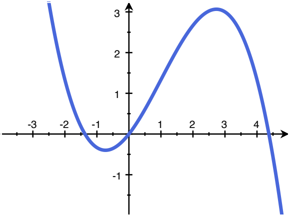

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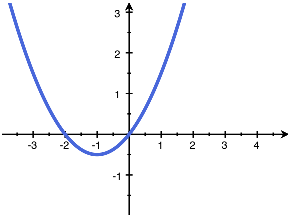

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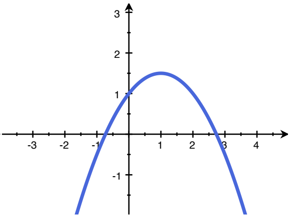

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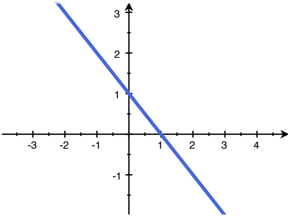

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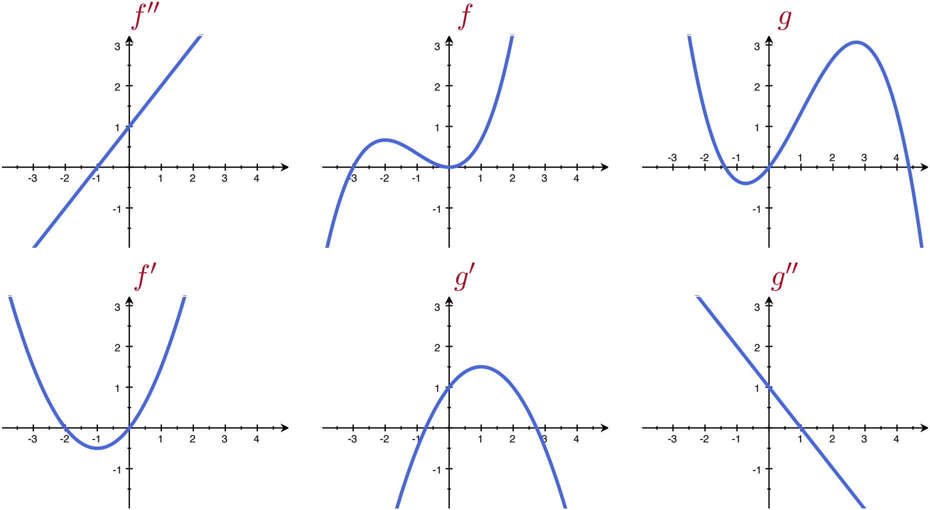
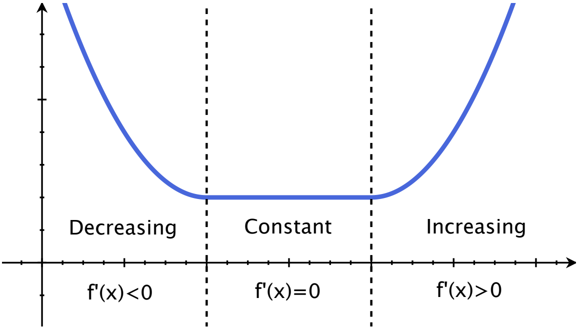


Review: Increasing/Decreasing

Suppose that f is continuous on $[a, b]$ and differentiable on the open interval (a, b) . Then

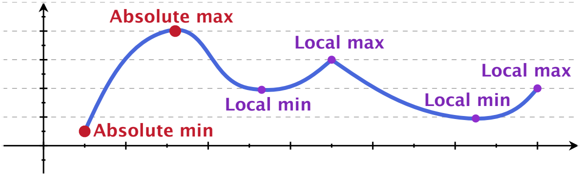
$$\text{If } f'(x) \text{ is } \left\{\begin{matrix}\text{positive}\\ \text{negative}\\ \text{zero}\end{matrix}\right\} \text{ for every } x \text{ in } (a,b) \text{ then } f \text{ is } \left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\\ \text{constant}\end{matrix}\right\} \text{ on } [a,b].$$

What it looks like:



Review: Extreme values

If f is continuous on a closed interval $[a, b]$, then there is a point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).



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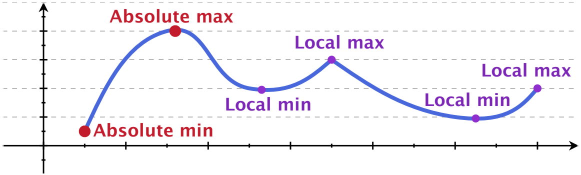
The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point c where $f'(c) = 0$ or $f'(c)$ is undefined.
   Vocab: If $f'(c)$ is undefined, c is also called a **singular point**.



Review: finding **absolute** min/max on **closed** intervals

1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

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x	$f(x)$	
1	-2	critical points
0	0	end points
2	2	

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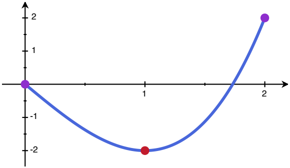
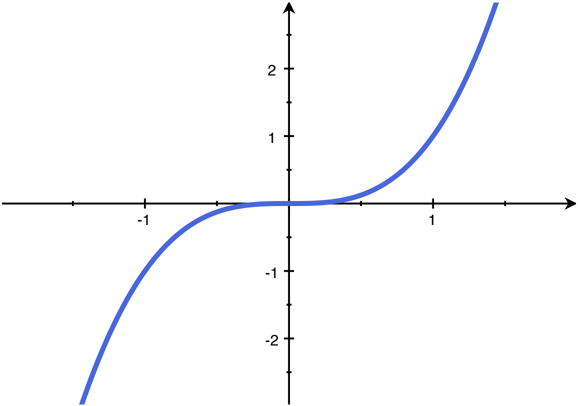


Finding **local** min/max on **any** intervals

Warning: Not all critical points are local minima or maxima:

Example: If $f(x) = x^3$, then $f'(x) = 3x^2$, and so $f'(0) = 0$:



Finding local extrema: The First Derivative Test

Suppose

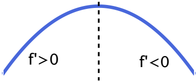
f is **continuous** on (a, b),

c is in (a, b) and is a **critical point** of $f(x)$, and

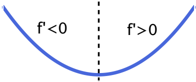
f is **differentiable** on (a, b) (except possibly at $x = c$)

Then the value $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from **positive** $\rightarrow$ **negative** at $x = c$, then $f(c)$ is a **local maximum**.



2. If $f'(x)$ changes from **negative** $\rightarrow$ **positive** at $x = c$, then $f(c)$ is a **local minimum**.



3. If $f'(x)$ doesn't change sign, then it's neither a min or a max.

Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

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$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$

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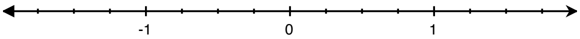

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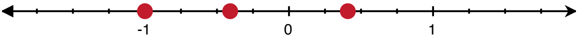

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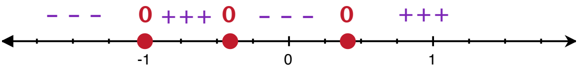

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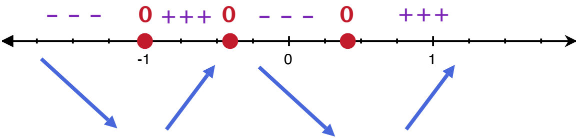

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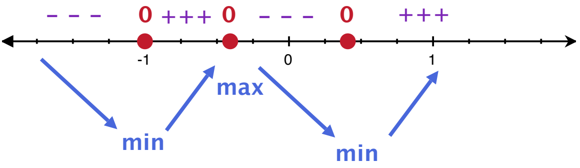

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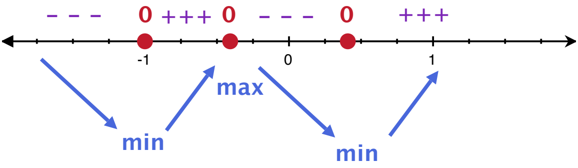
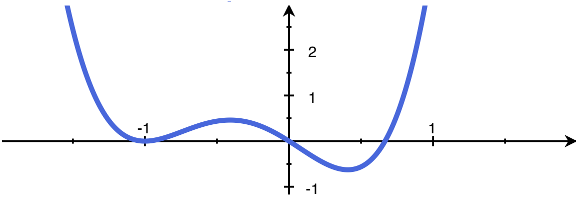




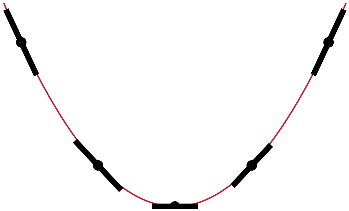
Example

Find the local extrema of $f(x) = \dfrac{x^4 + 1}{x^2}$ over the whole real line.

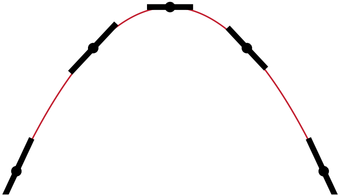
[Hint: Make sure to write the derivative like $f'(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.]

Concavity

Q. How can we measure when a function is concave up or down?



Concave up
$f'(x)$ is increasing



Concave down
$f'(x)$ is decreasing

Concavity

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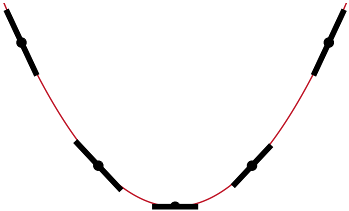
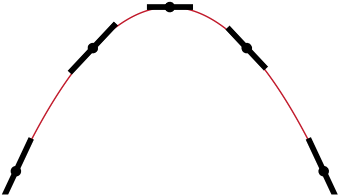


Concave up

$f'(x)$ is increasing

$\boxed{f''(x) > 0}$



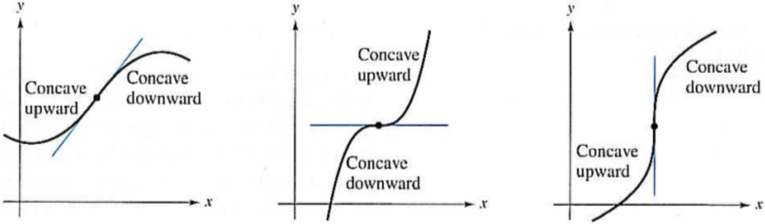
Concave down

$f'(x)$ is decreasing

$\boxed{f''(x) < 0}$

Concavity and Inflection Points

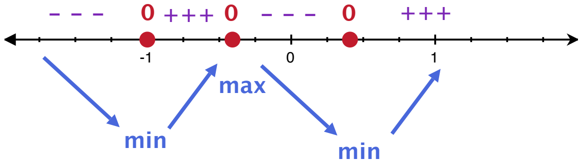
Definition: The function f has an **inflection point** at the point $x = c$ if $f(c)$ exists and the concavity changes at $x = c$ from up to down or vice versa.



Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.



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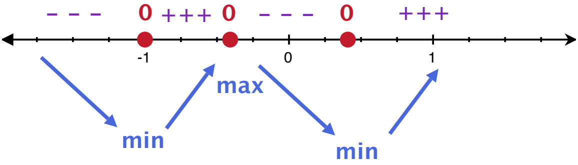

So

$$f''(x) = 36x^2 + 24x - 2 = (6x - (\sqrt{6} + 2))(6x + \sqrt{6} + 2)$$

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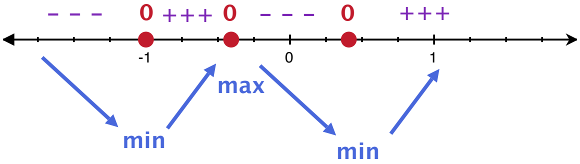

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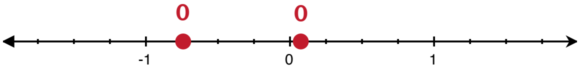

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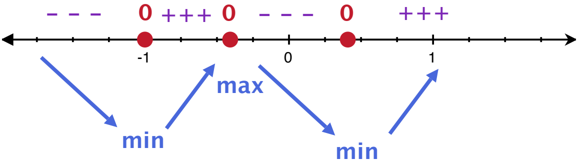

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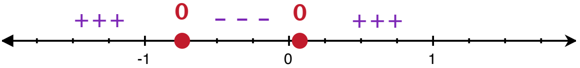

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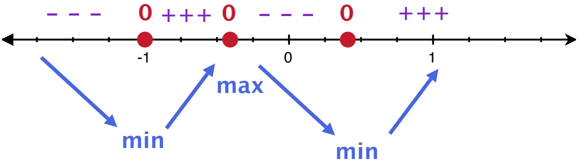

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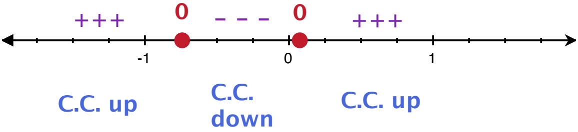
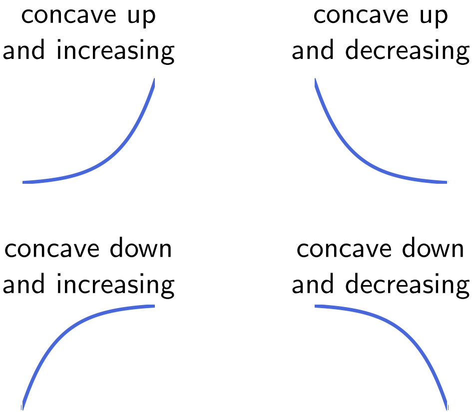
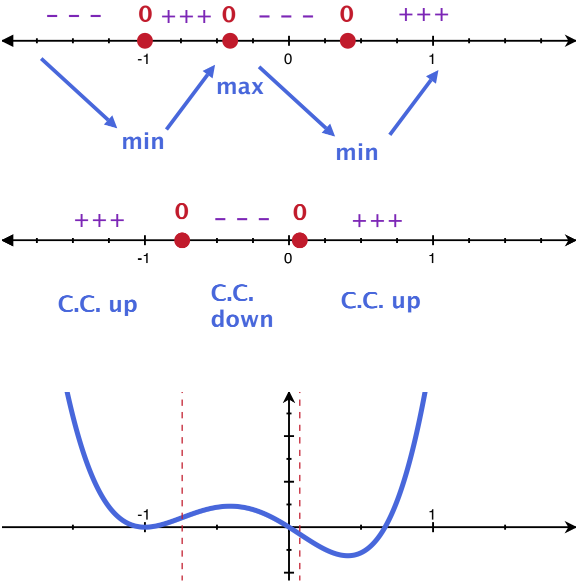


What the pieces look like



Putting it together



Last elements of graphing

Back to the example where $f(x) = \frac{x^4+1}{x^2}$:

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Step 1: Increasing/decreasing.

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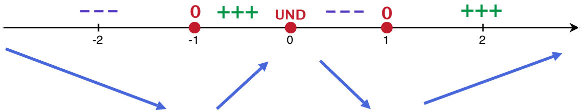

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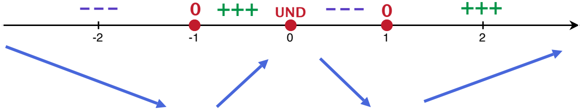

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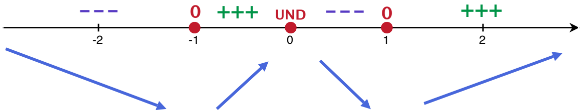

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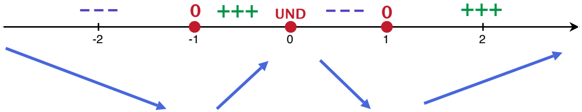

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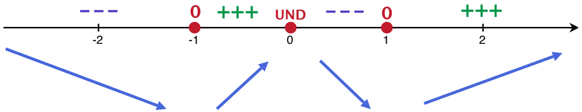
$$f''(x) = 2\frac{(4x^3)(x^3) - (x^4-1)(3x^2)}{x^6}$$

Last elements of graphing

Back to the example where $f(x) = \frac{x^4+1}{x^2}$:

Step 1: Increasing/decreasing.

We found $f'(x) = \frac{2(x^4-1)}{x^3} = 2\frac{(x^2+1)(x+1)(x-1)}{x^3}$



Step 2: Concavity.

$$f''(x) = 2\frac{(4x^3)(x^3) - (x^4-1)(3x^2)}{x^6} = 2\frac{4x^6 - 3x^6 + 3x^2}{x^6}$$

Last elements of graphing

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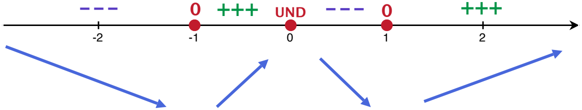

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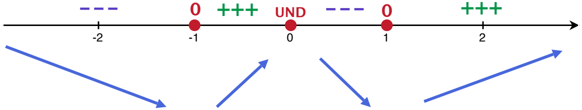

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Therefore $f''(x)$ is always positive, but is undef. at 0.

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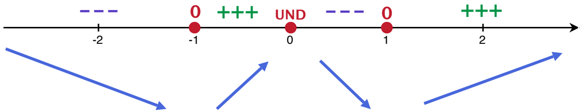

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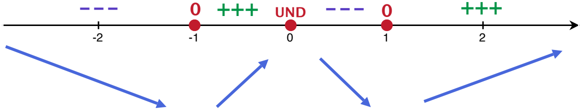
So $f(x)$ is always **concave up**

Last elements of graphing

Back to the example where $f(x) = \frac{x^4+1}{x^2}$:

Step 0: Domain. $f(x)$ is defined everywhere except $x = 0$
Step 1: Increasing/decreasing.

We found $f'(x) = \frac{2(x^4-1)}{x^3} = 2\frac{(x^2+1)(x+1)(x-1)}{x^3}$



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Therefore $f''(x)$ is always positive, but is undef. at 0.
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Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 4: Extreme behavior.

(a) What is $\lim_{x\to-\infty} f(x)$? What is $\lim_{x\to\infty} f(x)$?
Are there any *horizontal* asymptotes?

(b) For any hole in the domain $x = a$, what is $\lim_{x\to a^-} f(x)$?
$\lim_{x\to a^+} f(x)$? Are there any *vertical* asymptotes?

Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 4: Extreme behavior.

(a) What is $\lim_{x\to-\infty} f(x)$? What is $\lim_{x\to\infty} f(x)$?
Are there any *horizontal* asymptotes?

For example, $$\lim_{x\to-\infty} \frac{x^4+1}{x^2} = \infty \qquad \lim_{x\to\infty} \frac{x^4+1}{x^2} = \infty,$$

so there are no horizontal asymptotes.

(b) For any hole in the domain $x = a$, what is $\lim_{x\to a^-} f(x)$? $\lim_{x\to a^+} f(x)$? Are there any *vertical* asymptotes?

Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

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For example, $$\lim_{x\to 0^-} \frac{x^4+1}{x^2} = \infty \qquad \lim_{x\to 0^+} \frac{x^4+1}{x^2} = \infty,$$

so there is a two-sided vertical asymptote.

Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 5: Plot salient points.

(a) Find any roots of $f(x)$. (x-intercepts)

(b) Calculate $f(x)$ at $x = 0$ (y-intercept), critical points, and inflection points.

Last elements of graphing

$f(x) = \frac{x^4+1}{x^2}$ continued...

Step 5: Plot salient points.

(a) Find any roots of $f(x)$. (x-intercepts)

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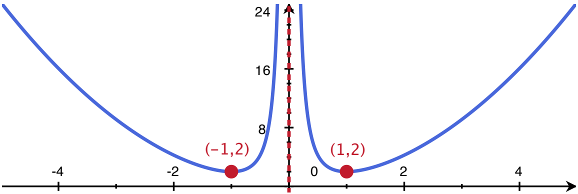
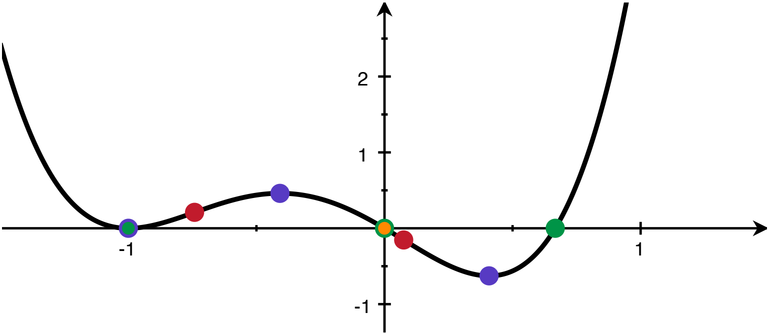


($f(x)$ doesn't have any roots, and doesn't have any inflection points)

Back to $3x^4 + 4x^3 - x^2 - 2x$

There are no vertical or horizontal asymptotes, and there are no points missing from the domain.

Points to mark:



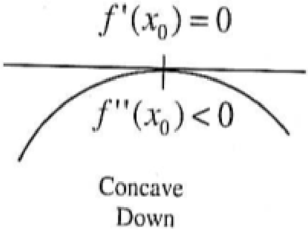
orange = y-intercept
green = roots
purple = critical points
red = inflection points.

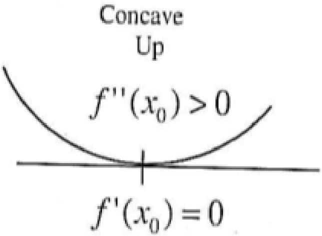
The second derivative test

Theorem

Let f be a function whose second derivative exists on an interval I containing x_0.

1. *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum.*
2. *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ is a local maximum.*



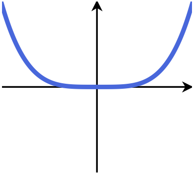


Warning: If $f'(x_0) = 0$ **and** $f''(x_0) = 0$, then the test **fails**! Use the first derivative test to decide instead.

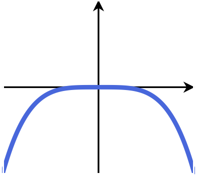
Why the 2^{nd} derivative test fails when $f''(x_0) = 0$

If $f'(x_0) = 0$ and $f''(x_0) = 0$, anything can happen!

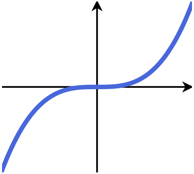
$f(x) = x^4$



$f(x) = -x^4$



$f(x) = x^3$



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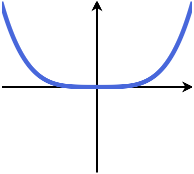
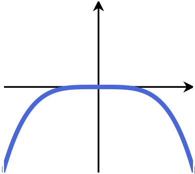


$f'(x) = 4x^3$
$f''(x) = 12x^2$

$f(x) = -x^4$



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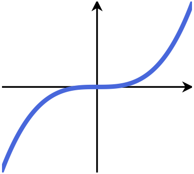
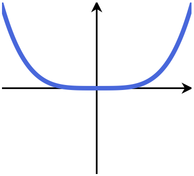


$f'(x) = 3x^2$
$f''(x) = 6x$

Why the 2^{nd} derivative test fails when $f''(x_0) = 0$

If $f'(x_0) = 0$ and $f''(x_0) = 0$, anything can happen!

$f(x) = x^4$



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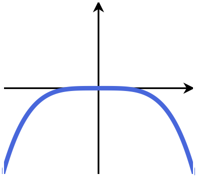
$f'(0) = 0$
$f''(0) = 0$

$f(x) = -x^4$

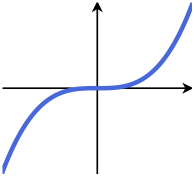


$f'(x) = -4x^3$
$f''(x) = -12x^2$

so

$f'(0) = 0$
$f''(0) = 0$

$f(x) = x^3$



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Why the 2^{nd} derivative test fails when $f''(x_0) = 0$

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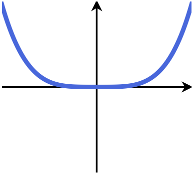
$f(x) = x^4$



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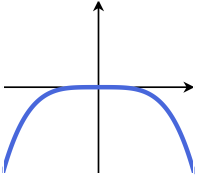

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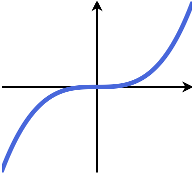

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so

$f'(0) = 0$
$f''(0) = 0$

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(The second derivative being zero just means the function is almost flat.)

Sketch graphs of the following functions:

1. $f(x) = -3x^5 + 5x^3$.
2. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

Instructions:

* Find any places where $f(x)$ is 0 or undefined.
* Calculate $f'(x)$ and find critical/singular points.
* Classify where $f'(x)$ is positive/negative, and therefore where $f(x)$ is increasing/decreasing.
* Calculate $f''(x)$, and find where it's 0 or undefined.
* Classify where $f''(x)$ is positive/negative, and therefore where $f(x)$ is concave up/down.
* Calculate $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ for anything where $f(a)$ is undefined.
* Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to see what the tails are doing.

Hint for 2: Always simplify as a fraction of polynomials after taking a derivative.