Going between graphs of functions and their derivatives:

Mean value theorem, Rolle's theorem, and intervals of increase and decrease

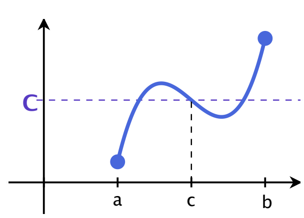
Recall: The *Intermediate* Value Theorem

Suppose f is continuous on a closed interval $[a, b]$.

If $\quad f(a) < C < f(b) \quad$ or $\quad f(a) > C > f(b),$

then there is at least one point c in the interval $[a, b]$ such that

$$f(c) = C.$$

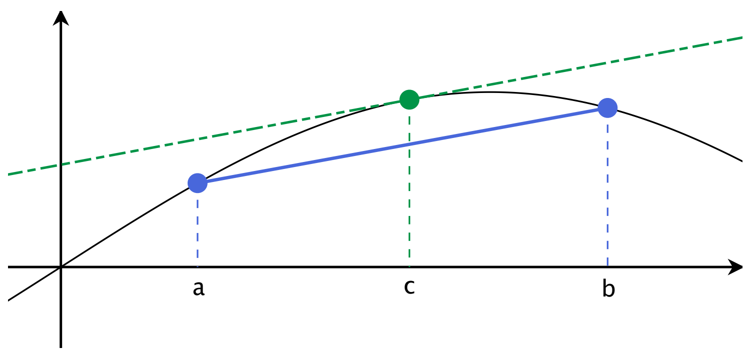


The *Mean* Value Theorem

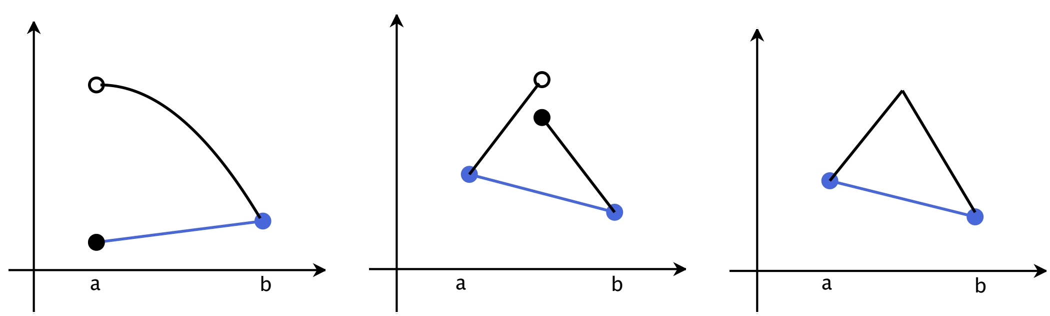
Theorem

Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that f' exists on the open interval (a, b). Then there exists a point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Bad examples



Discontinuity at an endpoint

Discontinuity at an interior point

No derivative at an interior point

Examples

Does the mean value theorem apply to $f(x) = |x|$ on $[-1, 1]$?

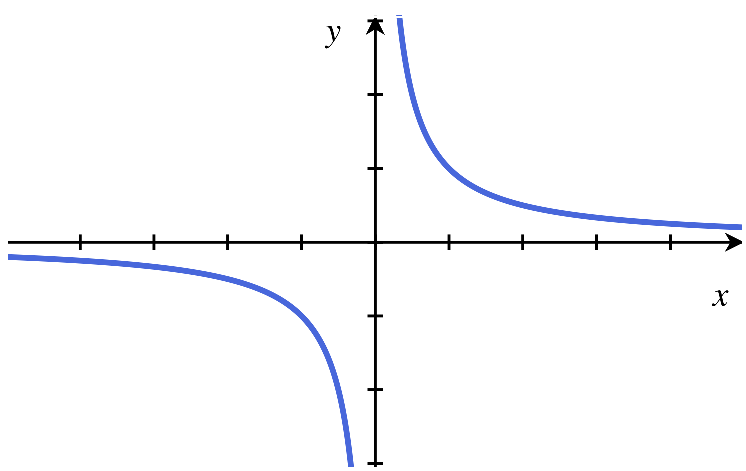
(No! Because $f(x)$ is not differentiable at $x = 0$.)

How about to $f(x) = |x|$ on $[1, 5]$?

(Yes! Because $f(x) = x$ on this domain, which is differentiable.)

Example

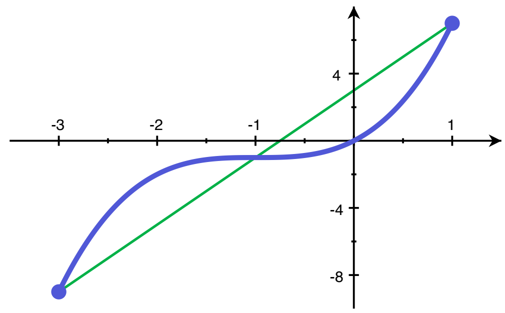
Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$?



ANY closed interval on the <u>domain</u>!

Example

Verify the conclusion of the Mean Value Theorem for the function $f(x) = (x+1)^3 - 1$ on the interval $[-3, 1]$.



Step 1: Check that the conditions of the MVT are met.

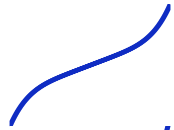
Step 2: Calculate the slope m of the line joining the two endpoints.

Step 3: Solve the equation $f'(x) = m$.

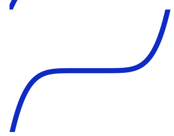
Intervals on increase/decrease

Formally,

f is *increasing* if
$f(x_1) < f(x_2)$ whenever $x_1 < x_2$.



f is *nondecreasing* if
$f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.



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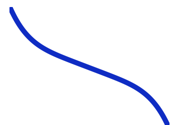
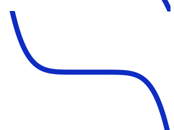
f is *decreasing* if
$f(x_1) > f(x_2)$ whenever $x_1 < x_2$.



f is *nonincreasing* if
$f(x_1) \geq f(x_2)$ whenever $x1 < x2$.



Sign of the derivative

If $f(x)$ is **increasing**, what is the sign of the derivative?
Look at the difference quotient:

$$\frac{f(x+h) - f(x)}{h}$$

The derivative is a two-sided limit, so we have two cases:

Case 1: h is positive.
So $x + h > x$, which implies $f(x+h) - f(x) > 0$.
So

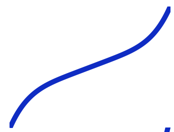
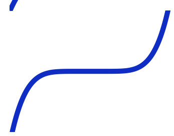
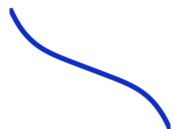
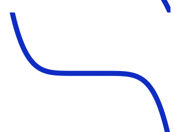
$$\frac{f(x+h) - f(x)}{h} > 0.$$

Case 2: h is negative.
So $x + h < x$, which implies $f(x+h) - f(x) < 0$.
So

$$\frac{f(x+h) - f(x)}{h} > 0.$$

So the difference quotient is positive!

Intervals on increase/decrease

Formally,		$\frac{f(x+h)-f(x)}{h}$	$\lim_{h\to 0} \sim$
f is *increasing* if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.		pos.	pos. or 0 (non-neg)
f is *nondecreasing* if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.		non-neg.	non-neg.
f is *decreasing* if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.		neg.	non-pos.
f is *nonincreasing* if $f(x_1) \geq f(x_2)$ whenever $x1 < x2$.		non-pos.	non-pos.

So we can calculate some of the "shape" of $f(x)$ by knowing when its derivative is positive, negative, and 0!

Example

On what interval(s) is the function $f(x) = x^3 + x + 1$ increasing or decreasing?

Step 1: Calculate the derivative.

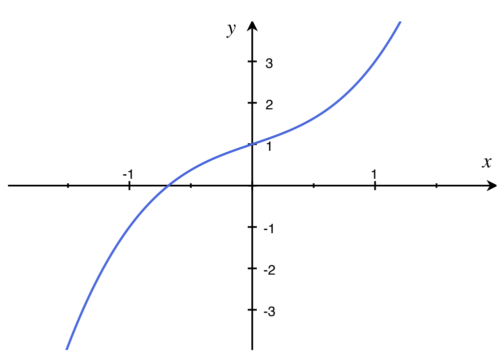
$$f'(x) = 3x^2 + 1$$

Step 2: Decide when the derivative is positive, negative, or zero.

$f'(x)$ is always positive!

Step 3: Bring that information back to $f(x)$.

$f(x)$ is always increasing!



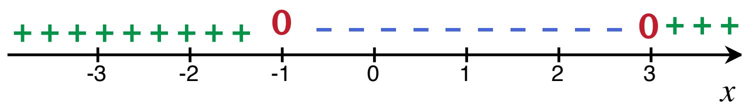
Example

Find the intervals on which the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing.

Step 1: Calculate the derivative.

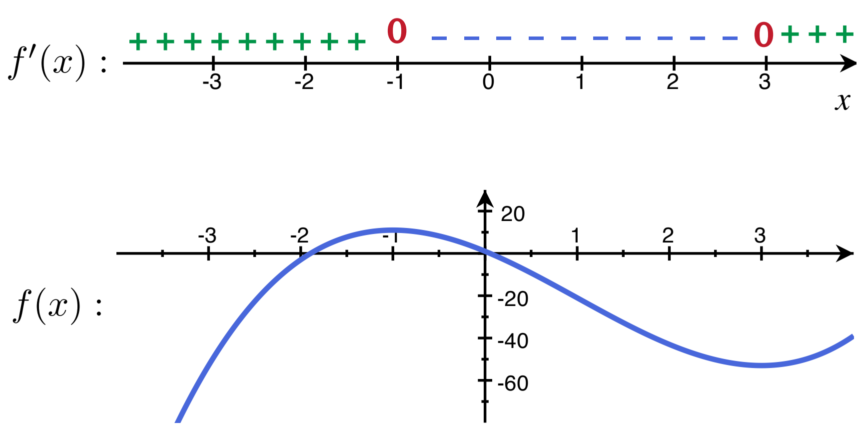
$$f'(x) = 6x^2 - 12x - 18 = 6(x-3)(x+1)$$

Step 2: Decide when the derivative is positive, negative, or zero.



Step 3: Bring that information back to $f(x)$.

$f(x)$ is increasing, then decreasing, then increasing.

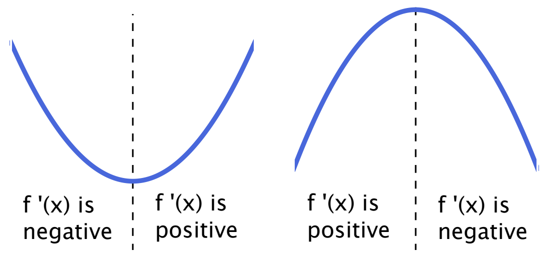


If f is continuous on a closed interval $[a, b]$, then there is at least one point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).

The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point c where $f'(c) = 0$ or $f(c)$ is undefined.

what's going on right before c?
what's going on right after c?



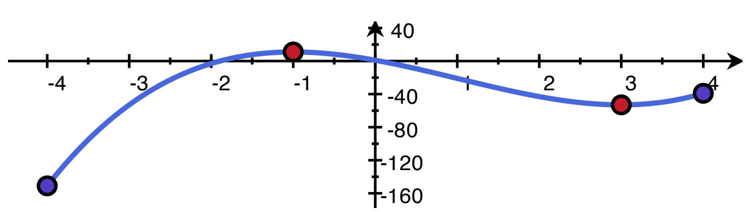
"local minimum" "local maximum"

Example

For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

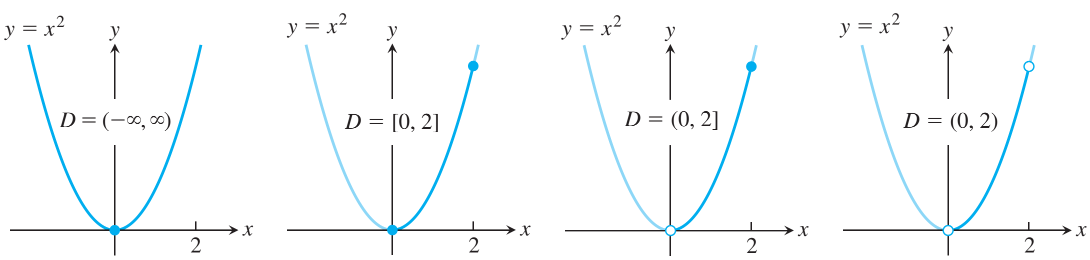
$$f'(x) = 6x^2 - 12x - 18 = 6(x - 3)(x + 1)$$

x	$f(x)$
-1	11
3	-53
-4	-151
4	-39



Absolute extrema depend on the domain!

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema



(a) abs min only (b) abs max and min (c) abs max only (d) no max or min

To compute absolute minima and maxima of $f(x)$ over a closed interval $[a, b]$:

1. compute the critical points c of $f(x)$ in $[a, b]$;
2. for each critical point c, compute $f(c)$; and
3. compute $f(a)$ and $f(b)$.

The absolute minima and maxima are the smallest and biggest numbers of those computed in steps 2 and 3.

To compute absolute minima and maxima of $f(x)$ over a open interval (a, b):

1. compute the critical points c of $f(x)$ in (a, b);
2. for each critical point c, compute $f(c)$; and
3. compute $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^+} f(b)$.

The absolute minima and maxima are the smallest and biggest numbers of those computed in step 2, *UNLESS* you got a smaller/bigger number in part 3, in which case no min/max exists.

Rolle's Theorem

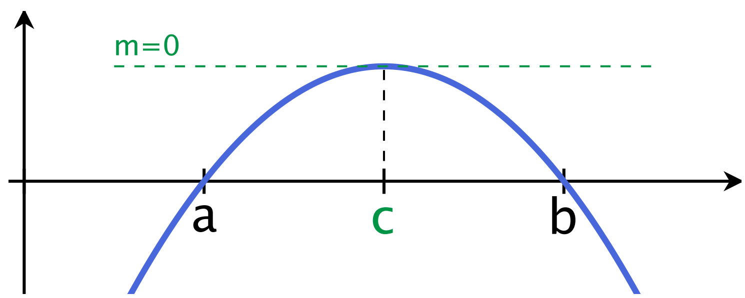
Theorem
Suppose that the function f is

continuous *on the closed interval* $[a, b]$,

differentiable *on the open interval* (a, b), *and*

a *and* b *are both* **roots** *of* f.

Then there is at least one point c *in* (a, b) *where* $f'(c) = 0$.



(In other words, if g didn't jump, then it had to turn around)

Again, the hypotheses matter!

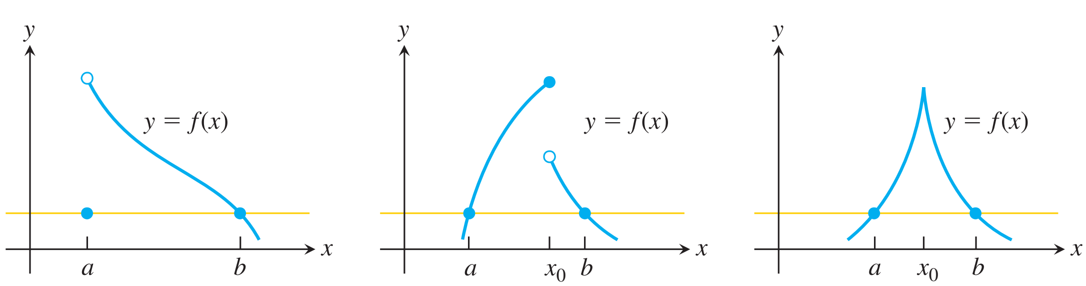
Rolle's Theorem. Suppose that the function f is

continuous on the closed interval $[a, b]$,

differentiable on the open interval (a, b), and

a and b are both **roots** of f.

Then there is at least one point c in (a, b) where $f'(c) = 0$.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

Example: Show that $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution: Use the intermediate value theorem, followed by Rolle's theorem!