# Going between graphs of functions and their derivatives:

Mean value theorem, Rolle's theorem, and intervals of increase and decrease
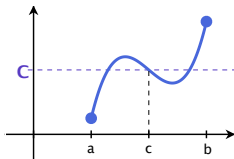
# Recall: The *Intermediate* Value Theorem

Suppose $f$ is continuous on a closed interval $[a, b]$.

If $\qquad f(a) < C < f(b) \qquad$ or $\qquad f(a) > C > f(b),$

then there is at least one point $c$ in the interval $[a, b]$ such that
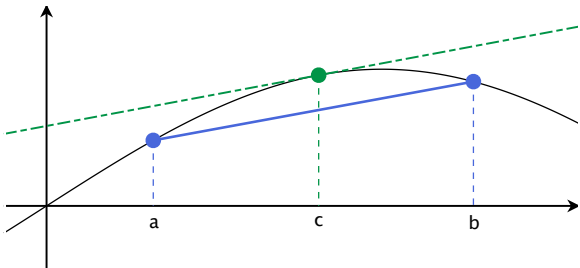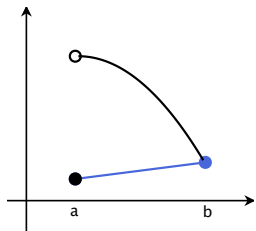
$$f(c) = C.$$

# The *Mean* Value Theorem

**Theorem**

*Suppose that $f$ is defined and continuous on a closed interval $[a, b]$, and suppose that $f'$ exists on the open interval $(a, b)$. Then there exists a point $c$ in $(a, b)$ such that*
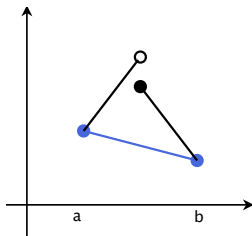
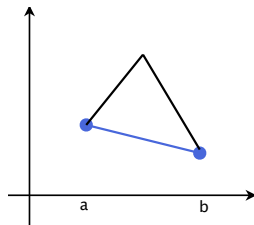$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

# Bad examples



Discontinuity at an endpoint

Discontinuity at an interior point

No derivative at an interior point

# Examples

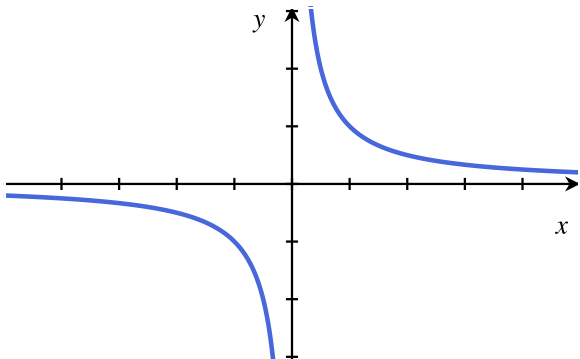Does the mean value theorem apply to $f(x) = |x|$ on $[-1, 1]$?

(No! Because $f(x)$ is not differentiable at $x = 0$.)

How about to $f(x) = |x|$ on $[1, 5]$?

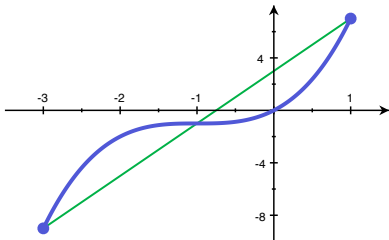(Yes! Because $f(x) = x$ on this domain, which is differentiable.)

# Example

Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$?



ANY closed interval on the <u>domain</u>!

# Example

Verify the conclusion of the Mean Value Theorem for the function $f(x) = (x+1)^3 - 1$ on the interval $[-3, 1]$.



**Step 1:** Check that the conditions of the MVT are met.

**Step 2:** Calculate the slope $m$ of the line joining the two endpoints.

**Step 3:** Solve the equation $f'(x) = m$.

# Intervals on increase/decrease

Formally,

$f$ is *increasing* if
$f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

$f$ is *nondecreasing* if
$f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.

$f$ is *decreasing* if
$f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

$f$ is *nonincreasing* if
$f(x_1) \geq f(x_2)$ whenever $x1 < x2$.

# Sign of the derivative

If $f(x)$ is **increasing**, what is the sign of the derivative?
Look at the difference quotient:

$$\frac{f(x+h) - f(x)}{h}$$

The derivative is a two-sided limit, so we have two cases:

**Case 1:** $h$ is positive.

So $x + h > x$, which implies $f(x+h) - f(x) > 0$.
So

$$\frac{f(x+h) - f(x)}{h} > 0.$$

**Case 2:** $h$ is negative.

So $x + h < x$, which implies $f(x+h) - f(x) < 0$.
So

$$\frac{f(x+h) - f(x)}{h} > 0.$$

So the difference quotient is positive!

# Intervals on increase/decrease

Formally,

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$f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

$f$ is *nonincreasing* if
$f(x_1) \geq f(x_2)$ whenever $x1 < x2$.

| | $\frac{f(x+h)-f(x)}{h}$ | $\lim_{h \to 0} \sim$ |
|---|---|---|
| | pos. | pos. or 0 (non-neg) |
| | non-neg. | non-neg. |
| | neg. | non-pos. |
| | non-pos. | non-pos. |

So we can calculate some of the "shape" of $f(x)$ by knowing when its derivative is positive, negative, and 0!

## Example

On what interval(s) is the function $f(x) = x^3 + x + 1$ increasing or decreasing?
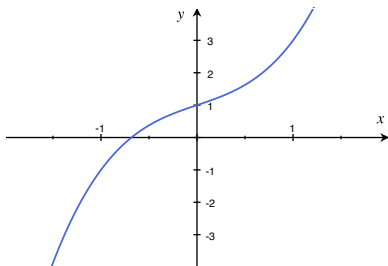
**Step 1:** Calculate the derivative.
$$f'(x) = 3x^2 + 1$$

**Step 2:** Decide when the derivative is positive, negative, or zero.
$$f'(x) \text{ is always positive!}$$

**Step 3:** Bring that information back to $f(x)$.
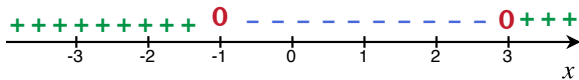$$f(x) \text{ is always increasing!}$$

# Example

Find the intervals on which the function
$f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing.
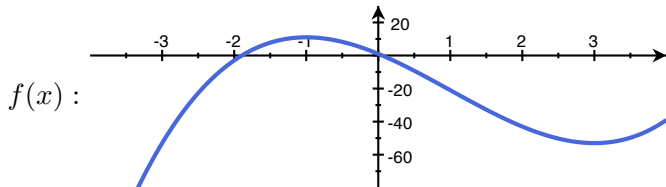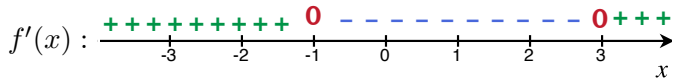
**Step 1:** Calculate the derivative.
$$f'(x) = 6x^2 - 12x - 18 = 6(x - 3)(x + 1)$$

**Step 2:** Decide when the derivative is positive, negative, or zero.

+ + + + + + + + + **0** – – – – – – – – – – **0** + + +

-3   -2   -1   0   1   2   3

$x$

**Step 3:** Bring that information back to $f(x)$.

$f(x)$ is increasing, then decreasing, then increasing.

$f'(x):$

$f(x):$

If $f$ is continuous on a closed interval $[a, b]$, then there is at least one point in the interval where $f$ is largest (**maximized**) and a point where $f$ is smallest (**minimized**).
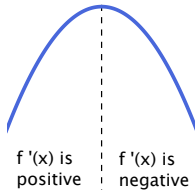
The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point $c$ where $f'(c) = 0$ or $f(c)$ is undefined.

what's going on right before $c$?
what's going on right after $c$?



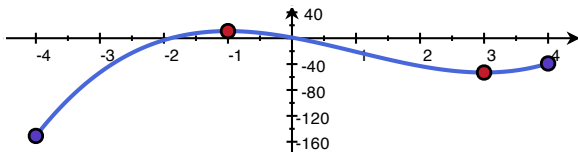| f '(x) is negative | f '(x) is positive | f '(x) is positive | f '(x) is negative |
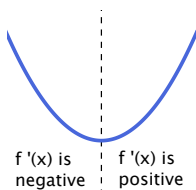
"local minimum"          "local maximum"

## Example

For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.
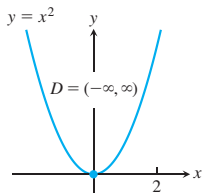
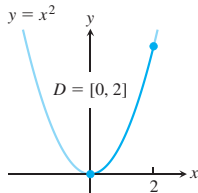$$f'(x) = 6x^2 - 12x - 18 = 6(x - 3)(x + 1)$$

| $x$ | $f(x)$ |
|-----|--------|
| $-1$ | $11$ |
| $3$ | $-53$ |
| $-4$ | $-151$ |
| $4$ | $-39$ |

# Absolute extrema depend on the domain!

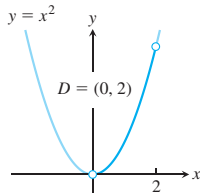| Function rule | Domain $D$ | Absolute extrema on $D$ |
|---|---|---|
| **(a)** $y = x^2$ | $(-\infty, \infty)$ | No absolute maximum<br>Absolute minimum of 0 at $x = 0$ |
| **(b)** $y = x^2$ | $[\,0, 2\,]$ | Absolute maximum of 4 at $x = 2$<br>Absolute minimum of 0 at $x = 0$ |
| **(c)** $y = x^2$ | $(0, 2\,]$ | Absolute maximum of 4 at $x = 2$<br>No absolute minimum |
| **(d)** $y = x^2$ | $(0, 2)$ | No absolute extrema |



(a) abs min only     (b) abs max and min     (c) abs max only     (d) no max or min

To compute absolute minima and maxima of $f(x)$ over a closed interval $[a, b]$:

1. compute the critical points $c$ of $f(x)$ in $[a, b]$;
2. for each critical point $c$, compute $f(c)$; and
3. compute $f(a)$ and $f(b)$.

The absolute minima and maxima are the smallest and biggest numbers of those computed in steps 2 and 3.

To compute absolute minima and maxima of $f(x)$ over a open interval $(a, b)$:

1. compute the critical points $c$ of $f(x)$ in $(a, b)$;
2. for each critical point $c$, compute $f(c)$; and
3. compute $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^+} f(b)$.

The absolute minima and maxima are the smallest and biggest numbers of those computed in step 2, *UNLESS* you got a smaller/bigger number in part 3, in which case no min/max exists.
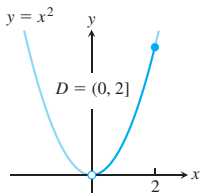
| Function rule | Domain $D$ | Absolute extrema on $D$ |
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| (a) $y = x^2$ | $(-\infty, \infty)$ | No absolute maximum<br>Absolute minimum of 0 at $x = 0$ |
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| (d) $y = x^2$ | $(0, 2)$ | No absolute extrema |

■
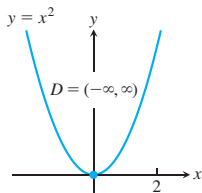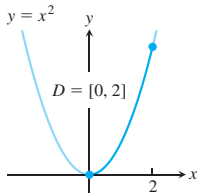
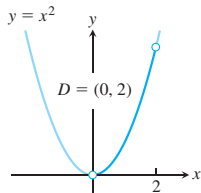

(a) abs min only    (b) abs max and min    (c) abs max only    (d) no max or min
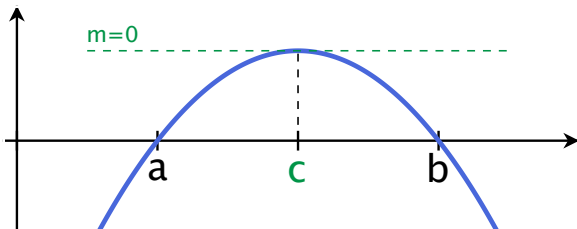
# Rolle's Theorem

**Theorem**

*Suppose that the function $f$ is*

    **continuous** *on the closed interval $[a, b]$,*

    **differentiable** *on the open interval $(a, b)$, and*

    $a$ *and* $b$ *are both* **roots** *of* $f$.

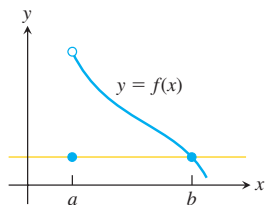*Then there is at least one point $c$ in $(a, b)$ where $f'(c) = 0$.*



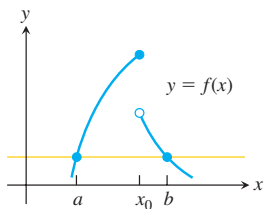(In other words, if $g$ didn't jump, then it had to turn around)

# Again, the hypotheses matter!

Rolle's Theorem. Suppose that the function $f$ is
**continuous** on the closed interval $[a, b]$,
**differentiable** on the open interval $(a, b)$, and
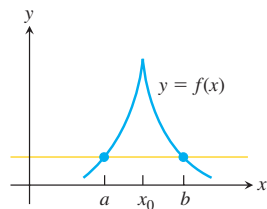$a$ and $b$ are both **roots** of $f$.

Then there is at least one point $c$ in $(a, b)$ where $f'(c) = 0$.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

Example: Show that $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution: Use the intermediate value theorem, followed by Rolle's theorem!