Logarithmic differentiation

Example: Calculate $\frac{dy}{dx}$ if $y = x^{\sin(x)}$

Problem: Both the base and the exponent have the variable in them! So we can't use

$$\frac{d}{dx}x^a = ax^{a-1} \qquad \text{or} \qquad \frac{d}{dx}a^x = \ln(a)a^x.$$

Fix: Take the log of both sides and use implicit differentiation:

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x) * \ln(x) \qquad \text{(using } \ln(a^b) = b\ln(a)\text{)}$$

Taking the derivative of both sides gives

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) + \sin(x)\frac{1}{x}$$

Then solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = y\left(\cos(x)\ln(x) + \sin(x)\frac{1}{x}\right) = \boxed{x^{\sin(x)}\left(\cos(x)\ln(x) + \sin(x)\frac{1}{x}\right)}.$$

Logarithmic differentiation

You try: Compute the derivatives of the following functions using logarithmic differentiation. Namely,

1. Let $y = f(x)$, and take the natural log of both sides
2. Use $\ln(a^b) = b\ln(a)$ and $\ln(ab) = \ln(a) + \ln(b)$ to expand.
3. Use implicit differentiation to compute $\frac{dy}{dx}$.
4. Plug back in $y = f(x)$, and simplify if necessary.

(a) $f(x) = 3^x$

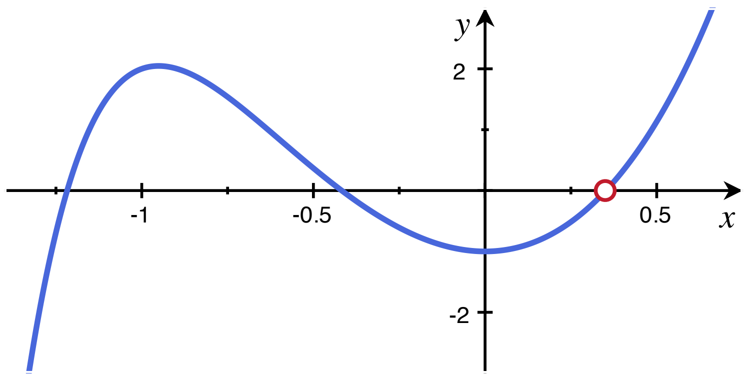
(b) $f(x) = x^x$

(c) $f(x) = \dfrac{(1+2x)^9(e^x + x^5)^{1/2}}{3x-1}$

Newton's Method for finding roots

Goal: Where is $f(x) = 0$?

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$



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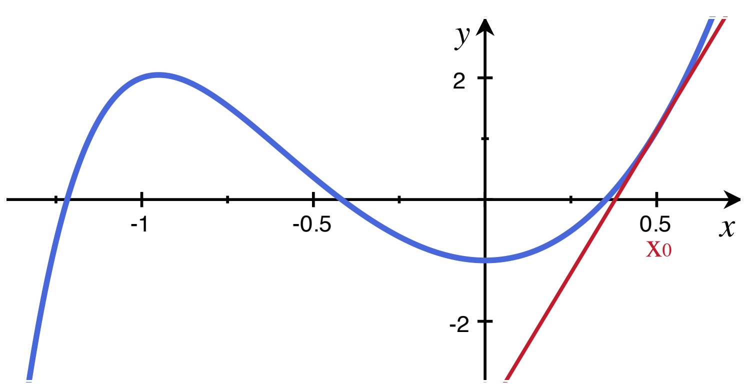

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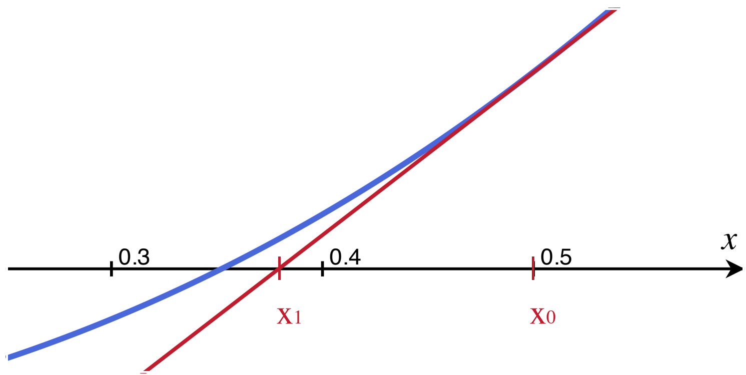


$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

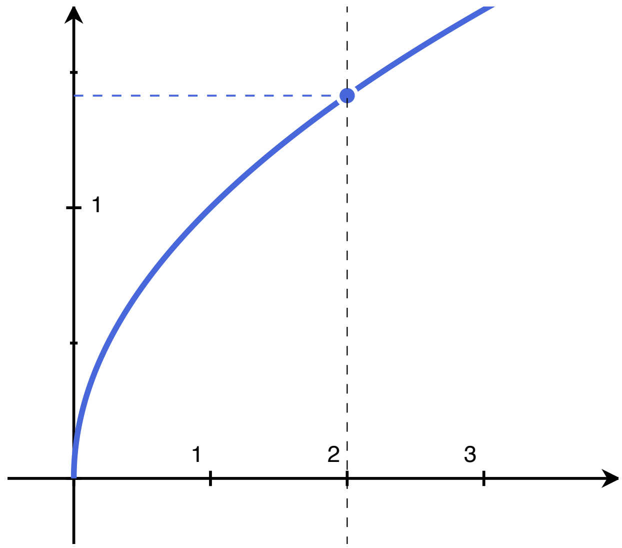
$$f'(x) = 7x^6 + 9x^2 + 14x$$

i	x_i	$f(x_i)$	$f'(x_i)$	tangent line	x-intercept
0	0.5	1.133	9.359	$y = 1.133 + 9.359(x - 0.5)$	0.379
1	0.379	0.170	6.619	$y = 0.170 + 6.619(x - 0.379)$	0.353
2	0.353	0.007	6.084	$y = 0.007 + 6.084(x - 0.353)$	0.352
3	0.352	0.00001	6.060	$y = 0.00001 + 6.060(x - 0.352)$	0.352

Linear approximations of functions

Goal: approximate functions

Example: approximate $\sqrt{2}$



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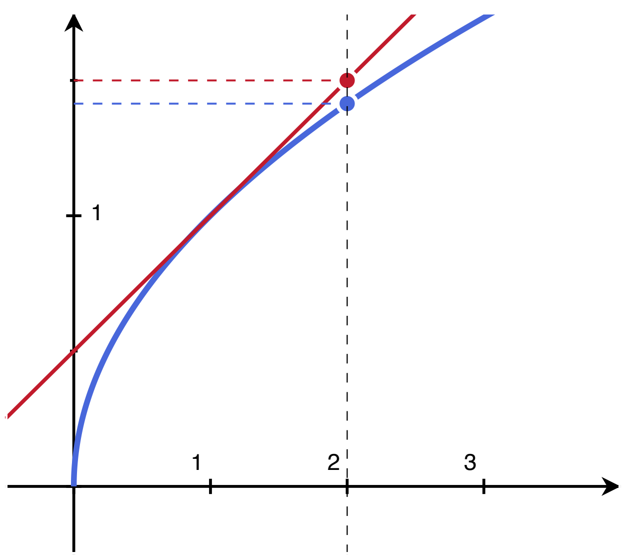


$$y = 1 + \frac{1}{2}(x-1)$$

$$\sqrt{2} \approx 1 + \frac{1}{2}(2-1) = 1.5 \qquad (\sqrt{2} = 1.414...)$$

Linear approximations

If $f(x)$ is differentiable at a, then the tangent line to $f(x)$ at $x = a$ is

$$y = f(a) + f'(a) * (x - a).$$

For values of x *near* a, then

$$f(x) \approx f(a) + f'(a) * (x - a).$$

This is the linearization (linear approximation) of $f(x)$ near $x = a$. We usually call the line $L(x)$.

Approximate $\sqrt{5}$:

Our last approximation told us

$$\sqrt{5} \approx L(5) = 1 + \frac{1}{2}(5 - 1) = 3$$

This isn't great... $(3^2 = 9)$

Better: Use the linearization about $x = 4$!

The tangent line at $x = 4$ is

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

so

$$\sqrt{5} \approx L(5) = 2 + \frac{1}{4}(5 - 4) = \boxed{2.25}$$

Better! $(2.25^2 = 5.0625)$

Aside: Find better approx's with higher derivatives. . .

The linearization (linear approximation) of $f(x)$ near a is **the** line which satisfies

$$L(a) = f(a) + f'(a)(a-a) = \boxed{f(a)}$$

and

$$L'(a) = \frac{d}{dx}\left(f(a) + f'(a)(x-a)\right) = \boxed{f'(a)}$$

A **better** approximation might be a quadratic polynomial $p_2(x)$ which **also** satisfies $p_2''(a) = f''(a)$:

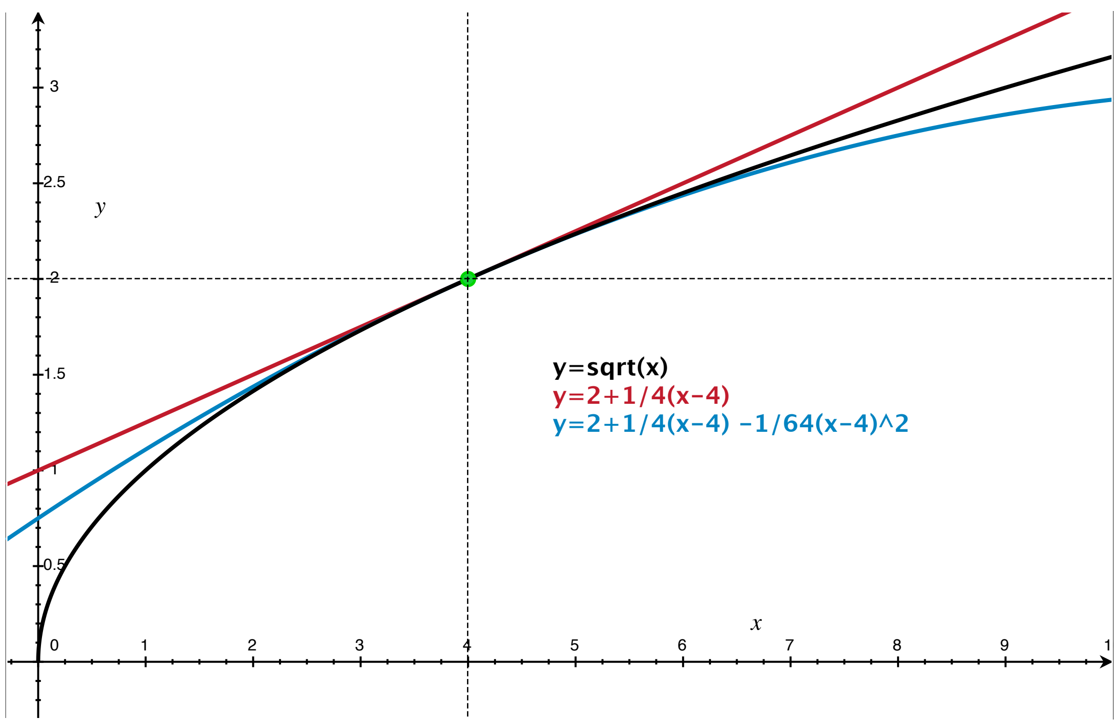
$$\boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2}$$

or a cubic polynomial $p_3(x)$ which also satisfies $p_3^{(3)}(a) = f^{(3)}(a)$:

$$\boxed{p_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{2*3}f^{(3)}(a)(x-a)^3}$$

and so on...

These approximations are called *Taylor polynomials*
(related to Taylor series, §10.8)



You try:

1. Compute the linearization of the following functions near the given point x_0.

(a) $f(x) = \sqrt{1+2x},\ x_0 = 4$

(b) $f(x) = x\cos(x),\ x_0 = 0$

2. What's wrong with computing a linearization of $f(x) = \sqrt{1+2x}$ at $x_0 = 3$?

3. If you wanted to approximate e^{3x-6} using a line, near what value(s) of x_0 could you get the best approximation with exact coefficients? Do it.

4. Use linearization to approximate $\sqrt{10}$, $\sqrt{15}$, and $\sqrt{20}$. For each answer, square your result to check how good your approximation was.