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# Logarithmic differentiation

You try: Compute the derivatives of the following functions using logarithmic differentiation. Namely,

1. Let $y = f(x)$, and take the natural log of both sides
2. Use $\ln(a^b) = b\ln(a)$ and $\ln(ab) = \ln(a) + \ln(b)$ to expand.
3. Use implicit differentiation to compute $\frac{dy}{dx}$.
4. Plug back in $y = f(x)$, and simplify if necessary.

---

(a) $f(x) = 3^x$

(b) $f(x) = x^x$

(c) $f(x) = \dfrac{(1+2x)^9(e^x+x^5)^{1/2}}{3x-1}$

# 3.11: Linearization ~~and Differentials~~

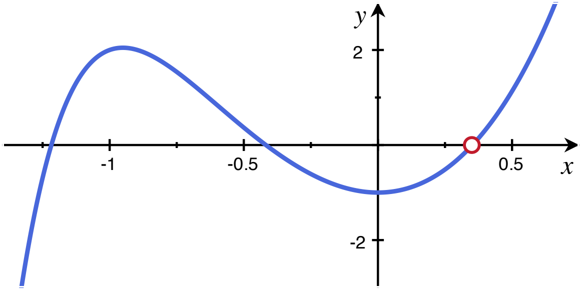
(Skip differentials)

A.K.A. Curves are tricky. Lines aren't.

# Newton's Method for finding roots

**Goal: Where is $f(x) = 0$?**

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$



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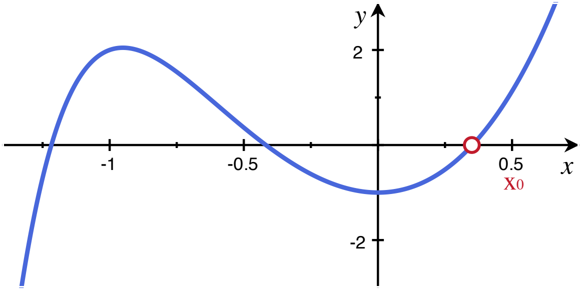

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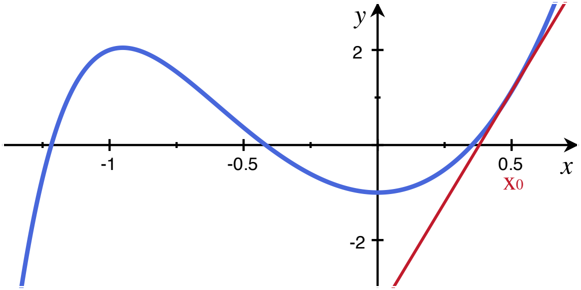

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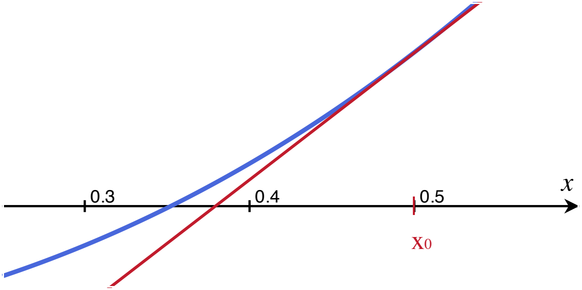

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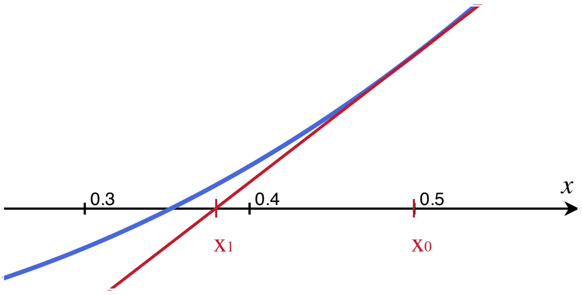

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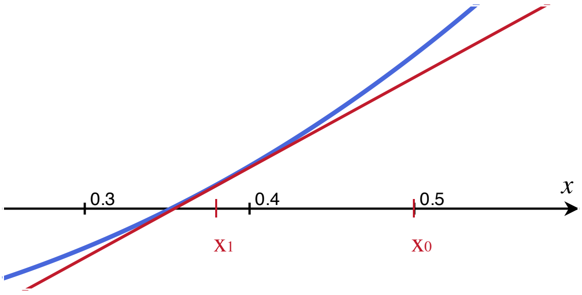

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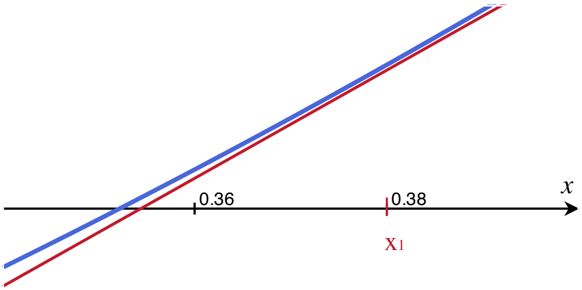

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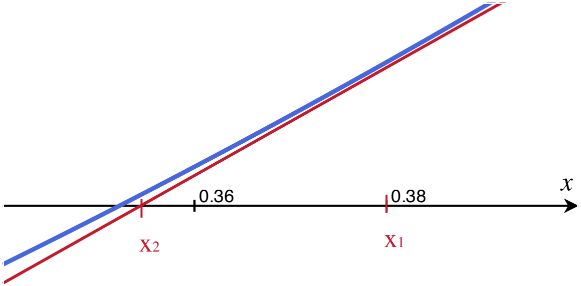

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

$$f'(x) = 7x^6 + 9x^2 + 14x$$

| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | tangent line | $x$-intercept |
|---|---|---|---|---|---|
| 0 | 0.5 | | | | |
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |

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| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
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|---|---|---|---|---|---|
| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | | | | | |
| 3 | | | | | |

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

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| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | | | | |
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| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | tangent line | $x$-intercept |
|---|---|---|---|---|---|
| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | 0.007 | 6.084 | $y = 0.007 + 6.084(x - 0.353)$ | 0.352 |
| 3 | | | | | |

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

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| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | tangent line | $x$-intercept |
|---|---|---|---|---|---|
| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | 0.007 | 6.084 | $y = 0.007 + 6.084(x - 0.353)$ | 0.352 |
| 3 | 0.352 | | | | |

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| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | 0.007 | 6.084 | $y = 0.007 + 6.084(x - 0.353)$ | 0.352 |
| 3 | 0.352 | 0.00001 | 6.060 | $y = 0.00001 + 6.060(x - 0.352)$ | 0.352 |

# Linear approximations of functions

**Goal: approximate functions**

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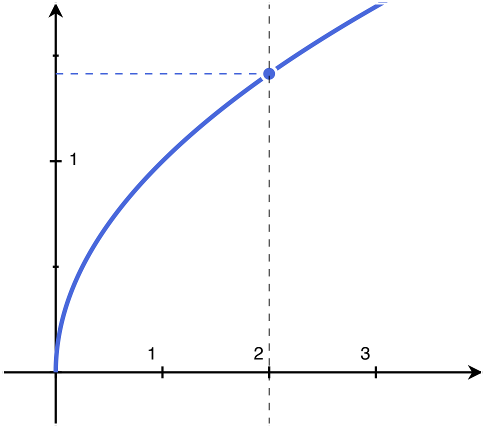
Example: approximate $\sqrt{2}$



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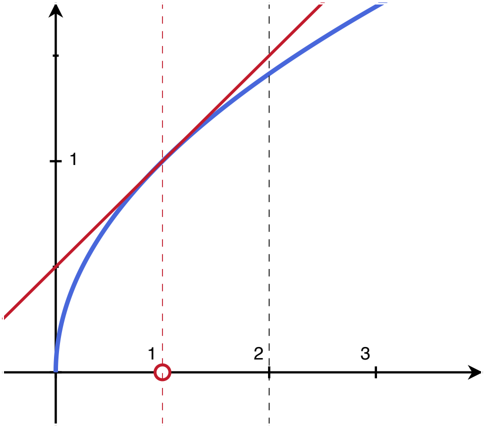


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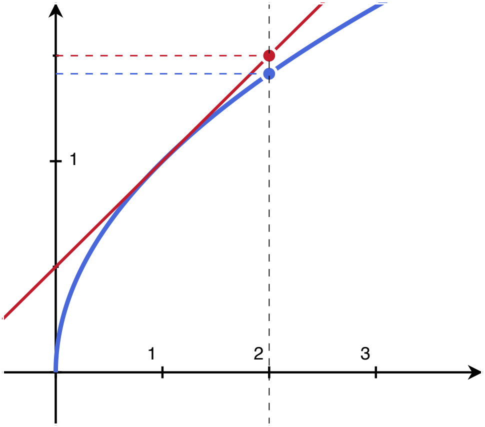


$$y = 1 + \frac{1}{2}(x - 1)$$
$$\sqrt{2} \approx 1 + \frac{1}{2}(2 - 1) = 1.5$$

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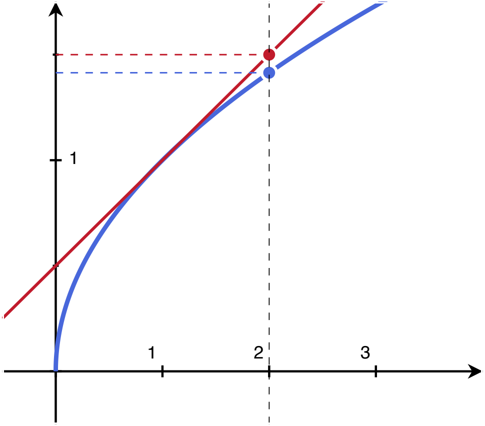


$$y = 1 + \frac{1}{2}(x-1)$$

$$\sqrt{2} \approx 1 + \frac{1}{2}(2-1) = 1.5 \qquad (\sqrt{2} = 1.414...)$$

# Linear approximations

If $f(x)$ is differentiable at $a$, then the tangent line to $f(x)$ at $x = a$ is

$$y = f(a) + f'(a) * (x - a).$$

For values of $x$ *near* $a$, then

$$f(x) \approx f(a) + f'(a) * (x - a).$$

This is the linearization (linear approximation) of $f(x)$ near $x = a$. We usually call the line $L(x)$.

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Our last approximation told us

$$\sqrt{5} \approx L(5) = 1 + \frac{1}{2}(5 - 1) = 3$$

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Better: Use the linearization about $x = 4$!

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The tangent line at $x = 4$ is

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

so

$$\sqrt{5} \approx L(5) = 2 + \frac{1}{4}(5 - 4) = \boxed{2.25}$$

Better! $(2.25^2 = 5.0625)$

# Aside: Find better approx's with higher derivatives...

The linearization (linear approximation) of $f(x)$ near $a$ is **the** line which satisfies

$$L(a) = f(a) + f'(a)(a - a) = \boxed{f(a)}$$

**and**

$$L'(a) = \frac{d}{dx}\big(f(a) + f'(a)(x - a)\big) = \boxed{f'(a)}$$

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A **better** approximation might be a quadratic polynomial $p_2(x)$ which **also** satisfies $p_2''(a) = f''(a)$:

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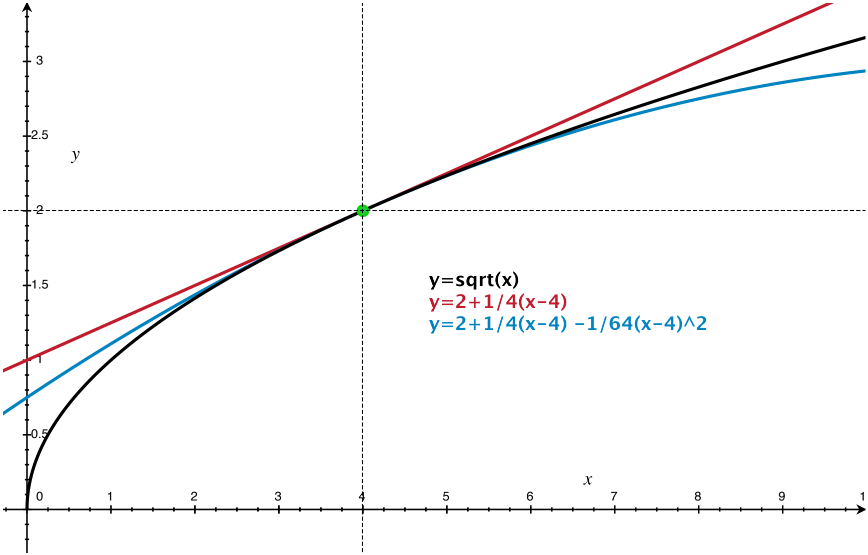
and so on...

These approximations are called *Taylor polynomials* (related to Taylor series, §10.8)



y
x
0
1
2
3
4
5
6
7
8
9
1
0.5
1
1.5
2
2.5
3
y=sqrt(x)
y=2+1/4(x-4)
y=2+1/4(x-4) -1/64(x-4)^2

# You try:

1. Compute the linearization of the following functions near the given point $x_0$.

(a) $f(x) = \sqrt{1+2x},\ x_0 = 4$

(b) $f(x) = x\cos(x),\ x_0 = 0$

2. What's wrong with computing a linearization of $f(x) = \sqrt{1+2x}$ at $x_0 = 3$?

3. If you wanted to approximate $e^{3x-6}$ using a line, near what value(s) of $x_0$ could you get the best approximation with exact coefficients? Do it.

4. Use linearization to approximate $\sqrt{10}$, $\sqrt{15}$, and $\sqrt{20}$. For each answer, square your result to check how good your approximation was.