Reviewing the worksheet

On the worksheet, you learned how to compute the derivatives of inverse functions, such as $\ln(x)$, $\arcsin(x)$, etc.

You used implicit differentiation to calculate the derivatives

1. $\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$
2. $\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))}$

and the rule

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to check your answers, and then to calculate the derivatives of the other inverse trig functions:

1. $\frac{d}{dx}\arccos(x) = -\left(\frac{1}{\sin(\arccos(x)}\right)$
2. $\frac{d}{dx}\text{arccot}(x) = -\left(\frac{1}{\csc^2(\text{arccot}(x))}\right)$
3. $\frac{d}{dx}\text{arcsec}(x) = \frac{1}{\sec(\text{arcsec}(x))\tan(\text{arcsec}(x))}$
4. $\frac{d}{dx}\text{arccsc}(x) = -\left(\frac{1}{\csc(\text{arccsc}(x))\cot(\text{arccsc}(x))}\right)$

Using implicit differentiation to calculate $\frac{d}{dx}\arcsin(x)$

$$\boxed{\text{If } y = \arcsin(x) \text{ then } x = \sin(y).}$$

Take $\frac{d}{dx}$ of both sides of $x = \sin(y)$:

Left hand side: $$\frac{d}{dx}x = 1$$

Right hand side: $$\frac{d}{dx}\sin(y) = \cos(y) * \frac{dy}{dx} = \cos(\arcsin(x)) * \frac{dy}{dx}$$

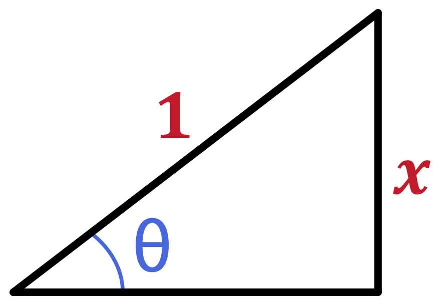
So

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))}.}$$

Simplifying $\cos(\arcsin(x))$

Call $\arcsin(x) = \theta$.

$\sin(\theta) = x$



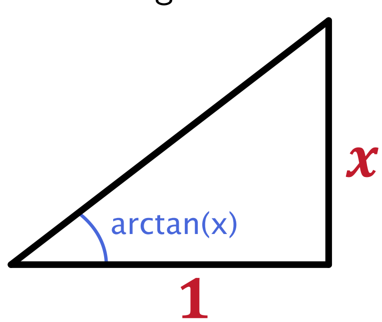
Key: This is a simple triangle to write down whose angle θ has $\sin(\theta) = x$

Calculating $\frac{d}{dx}\arctan(x)$.

We found that

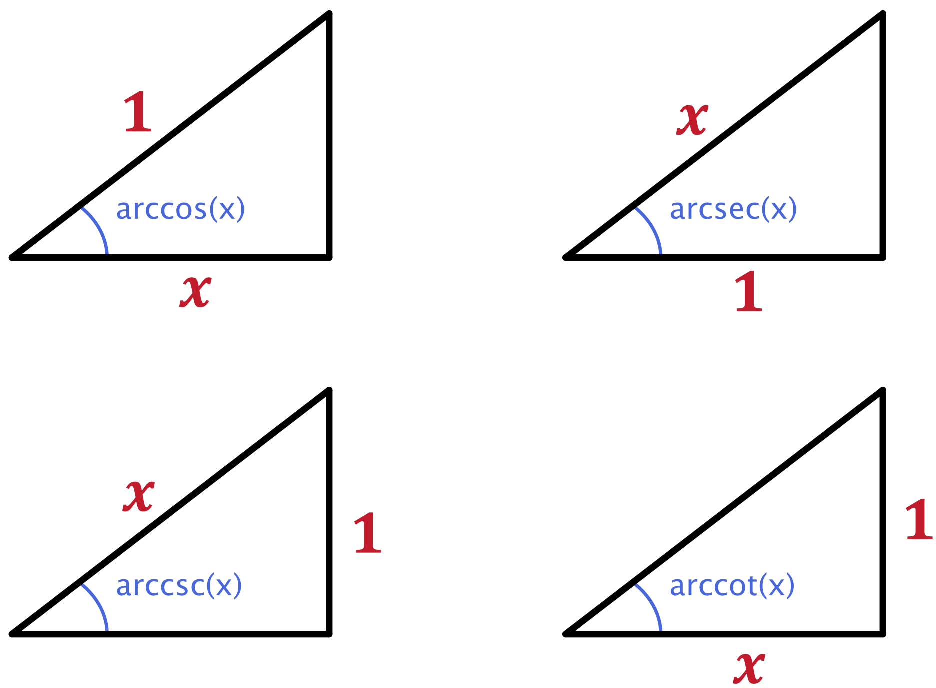
$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \left(\frac{1}{\sec(\arctan(x))}\right)^2$$

Simplify this expression using



$$\frac{dy}{dx} = \left(\frac{1}{\sec(\arctan(x))}\right)^2 = \frac{1}{1+x^2}$$

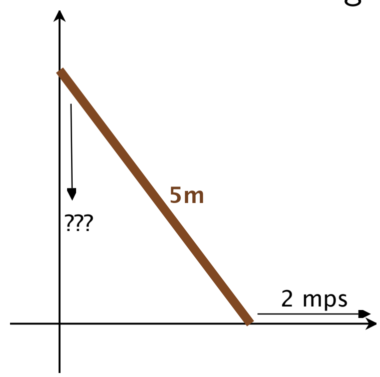
To simplify the rest, use the triangles



Today: Related rates.

Example:

Suppose you has a 5m ladder resting against a wall.



Move the base out at 2 m/s

How fast does the top move down the wall?

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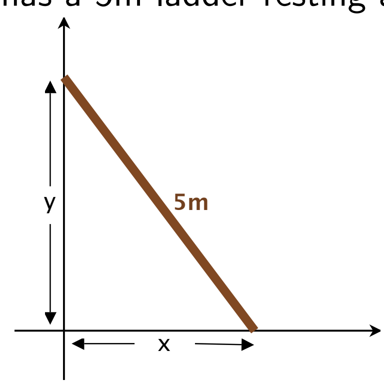


Move the base out at 2 m/s: $\frac{dx}{dt} = 2$

How fast does the top move down the wall? $\frac{dy}{dt} = ??$

To solve, we need to relate the variables: $x^2 + y^2 = 5^2$

$$0 \leq x \leq 5$$

Problem:
If $x^2 + y^2 = 5^2$ for $0 \leq x \leq 5$, and $\frac{dx}{dt} = 2$, what is $\frac{dy}{dt}$?

Differentiate:

$$\begin{aligned} 0 = \frac{d}{dt}5^2 &= \frac{d}{dt}\left(x^2 + y^2\right) \\ &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2x * 2 + 2\left(\sqrt{25 - x^2}\right)\frac{dy}{dt} \end{aligned}$$

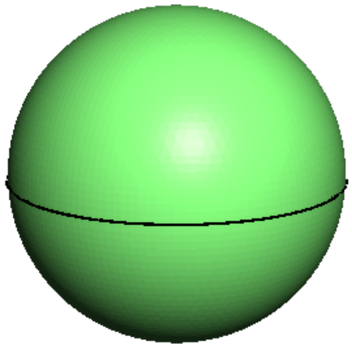
So

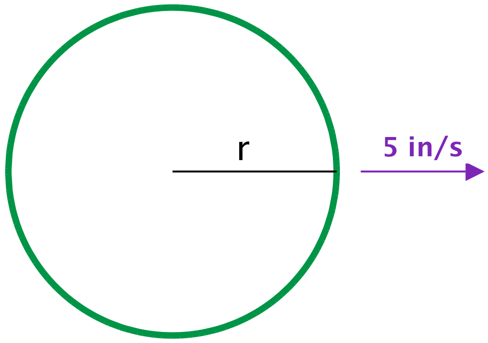
$$\boxed{\frac{dy}{dt} = \frac{-2x}{\sqrt{25 - x^2}}}$$

Notice:
(1) $\frac{dy}{dt} < 0$ (y is decreasing) and (2) $\lim_{x \to 5^-} \frac{dy}{dt} = -\infty$

Example

Suppose you have a sphere whose radius is growing at a rate of 5in/s. How fast is the volume growing when the radius is 3in?



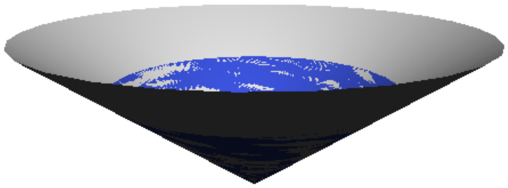


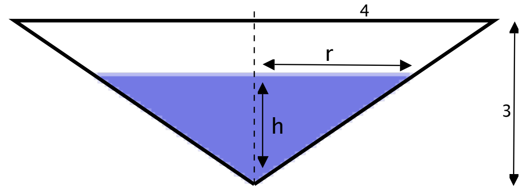
Relating equation: $V = \frac{4}{3}\pi r^3$

Take a derivative: $\frac{dV}{dt} = \frac{4}{3}\pi * 3r^2 * \frac{dr}{dt}$

Substitute in the known values:
$\frac{dV}{dt}\Big|_{r=3} = 4\pi * 3^2 * 5 = \boxed{4 * 9 * 5\pi \text{ in}^3/\text{s}}$

Take an upside-down cone-shaped bowl, with a radius of 4in at the top and a total height of 3in fill it with water at a rate of 1/2 in^3/min. How fast is the height of water increasing when h=2in?





Volume of a cone: $V = \dfrac{\pi}{3} R^2 H$

Volume of a water: $V = \dfrac{\pi}{3} r^2 h$

Relate r and h: $r/h = 4/3$ so $r = \frac{4}{3}h$

Finally, equation to differentiate: $V = \dfrac{\pi}{3}\left(\dfrac{4}{3}h\right)^2 h = \dfrac{\pi 16}{27} h^3$

$$\frac{1}{2} = \frac{dV}{dt} = \frac{\pi 16}{27} * 3h^2 \frac{dh}{dt} = \frac{\pi 16}{9}(2)^2 \left.\frac{dh}{dt}\right|_{h=2}$$

So $\boxed{\left.\frac{dh}{dt}\right|_{h=2} = \frac{9}{128\pi}}$

Strategy:

1. Find an equation which relates the functions you need.
   (a) Draw pictures!
   (b) Sometimes you'll have to reduce the number of variables/functions to get it down to
      (i) the function from the rate you know,
      (ii) the function from the rate you want, and
      (iii) maybe the variable from the rate you know and want (t in the last 3 examples).
2. Take a derivative using implicit differentiation.
3. Plug in the values you know.
4. Solve for the rate you want.

One more example: (from extra problems—see website)

10. A boat is pulled into a dock by a rope attached to the bow of the boat a passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s how fast is the boat approaching the dock when it is 8 m from the dock?

On your own: (from extra problems—see website)

5. A balloon which always remains spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15cm.

11. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

9. A lighthouse is on a small island 3 km away from the nearest point P on a straight shoreline and its light turns four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P? [hint: 4rpm means that some angle is changing at $4 * 2\pi$ radians per minute]