## Using implicit differentiation for good: Inverse functions.

Warmup: Calculate $\frac{dy}{dx}$ if

1. $e^y = xy$

2. $\cos(y) = x + y$

**Every time:**
(1) Take $\frac{d}{dx}$ of both sides.
(2) Add and subtract to get the $\frac{dy}{dx}$ terms on one side and everything else on the other.
(3) Factor out $\frac{dy}{dx}$ and divide both sides by its coefficient.

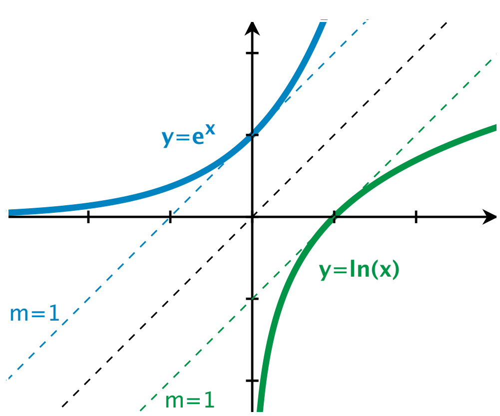
## The Derivative of $y = \ln x$

Remember:

(1) $y = e^x$ has a slope through the point (0,1) of 1.

(2) The natural log is the inverse function of $e^x$ , so

$$y = \ln x \quad \Leftrightarrow \quad e^y = x$$



## The Derivative of $y = \ln x$

To find the derivative of $\ln(x)$, use implicit differentiation!
Rewrite

$$y = \ln x \quad \text{as} \quad e^y = x$$

Take a derivative of both sides of $e^y = x$ to get

$$\frac{dy}{dx}e^y = 1 \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{e^y}$$

**Problem:** We asked "what is the derivative of $\ln(x)$?" and got back and answer with $y$ in it!
**Solution:** Substitute back!

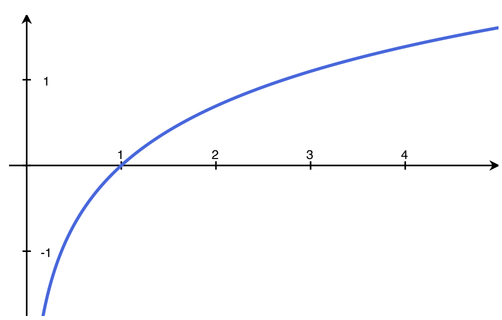
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

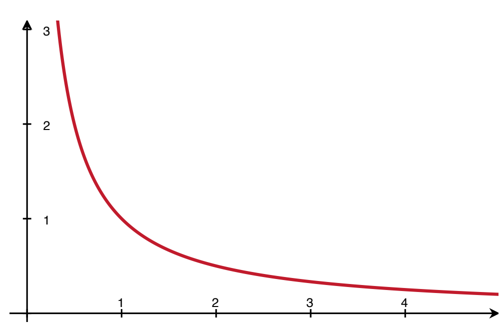
## Does it make sense?

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

$f(x) = \ln(x)$




$f(x) = \frac{1}{x}$



# Examples

Calculate

1. $\frac{d}{dx}\ln x^2$

2. $\frac{d}{dx}\ln(\sin(x^2))$

3. $\frac{d}{dx}\log_3(x)$

   [hint: $\log_a x = \frac{\ln x}{\ln a}$]

# Quick tip: Logarithmic differentiation

Example: Calculate $\frac{dy}{dx}$ if $y = x^{\sin(x)}$

Problem: Both the base and the exponent have the variable in them! So we can't use

$$\frac{d}{dx}x^a = ax^{a-1} \qquad \text{or} \qquad \frac{d}{dx}a^x = \ln(a)a^x.$$

Fix: Take the log of both sides and use implicit differentiation:

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x) * \ln(x) \qquad \text{(using } \ln(a^b) = b\ln(a)\text{)}$$

Taking the derivative of both sides gives

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) + \sin(x)\frac{1}{x}$$

Then solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = y\left(\cos(x)\ln(x) + \sin(x)\frac{1}{x}\right) = \boxed{x^{\sin(x)}\left(\cos(x)\ln(x) + \sin(x)\frac{1}{x}\right)}.$$

## Back to inverses

In the case where $y = \ln(x)$, we used the fact that $\ln(x) = f^{-1}(x)$, where $f(x) = e^x$, and got

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}.$$

In general, calculating $\frac{d}{dx}f^{-1}(x)$:

(1) Rewrite $y = f^{-1}(x)$ as $f(y) = x$.

(2) Use implicit differentiation:

$$f'(y) * \frac{dy}{dx} = 1 \qquad \text{so} \qquad \boxed{\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}}.$$

## Examples

Just to check, use the rule

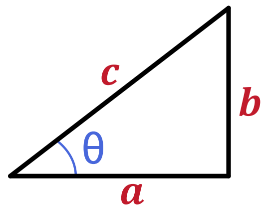
$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to calculate

1. $\frac{d}{dx}\ln(x)$ (the inverse of $e^x$)
   In the notation above, $f^{-1}(x) = \ln(x)$ and $f(x) = e^x$.
   We'll also need $f'(x) = e^x$. So
   $$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}} \quad ☺$$
2. $\frac{d}{dx}\sqrt{x}$ (the inverse of $x^2$)
   In the notation above, $f^{-1}(x) = \sqrt{x}$ and $f(x) = x^2$.
   We'll also need $f'(x) = 2x$. So
   $$\boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2 * (\sqrt{x})}} \quad ☺$$

In general:

arc___( - ) takes in a ratio and spits out an angle:




$$\cos(\theta) = a/c \quad \text{so} \quad \arccos(a/c) = \theta$$

$$\sin(\theta) = b/c \quad \text{so} \quad \arcsin(b/c) = \theta$$

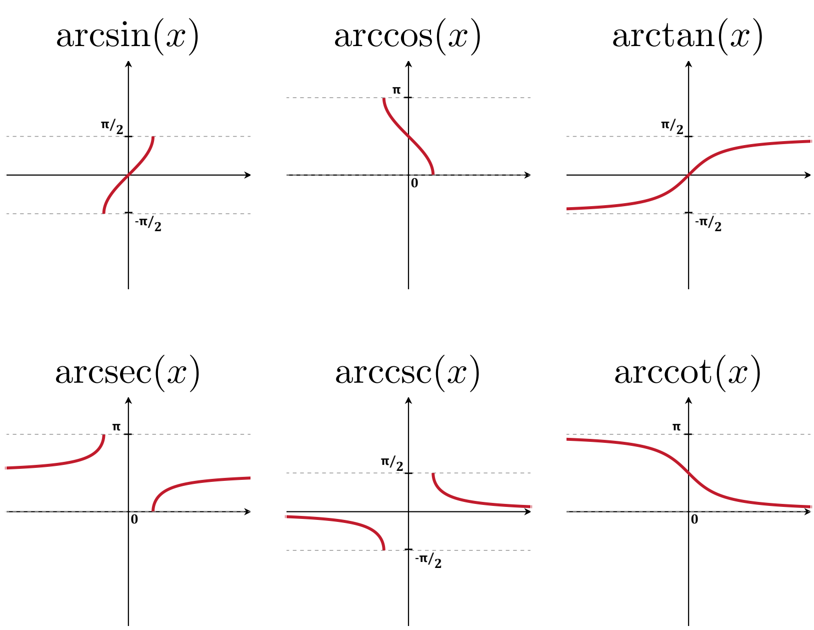
$$\tan(\theta) = b/a \quad \text{so} \quad \arctan(b/a) = \theta$$

**Domain problems:**

$$\sin(0) = 0, \qquad \sin(\pi) = 0, \qquad \sin(2\pi) = 0, \qquad \sin(3\pi) = 0, \ldots$$

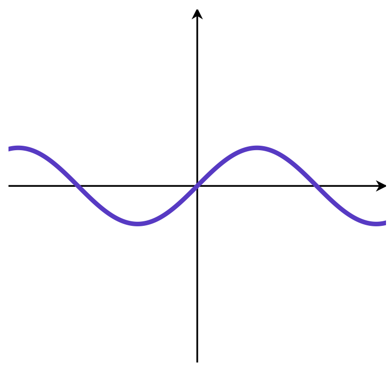
So which is the right answer to $\arcsin(0)$, really?

## Graphs



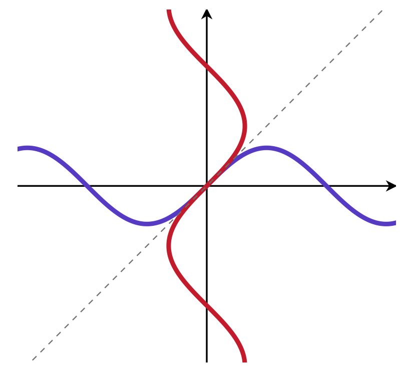
## Domain/range

$y = \sin(x)$



## Domain/range

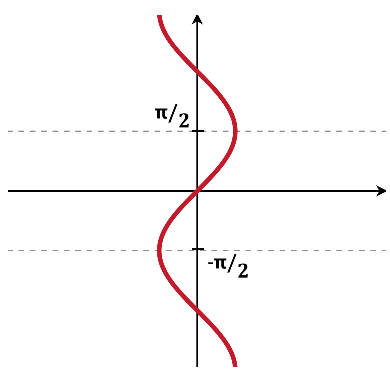
$y = \sin(x)$
$y = \arcsin(x)$



Domain: $-1 \leq x \leq 1$

## Domain/range

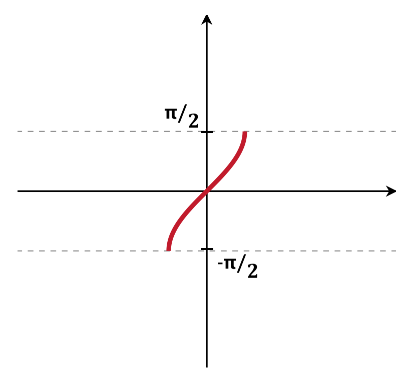
$y = \arcsin(x)$




Domain: $-1 \leq x \leq 1$

## Domain/range

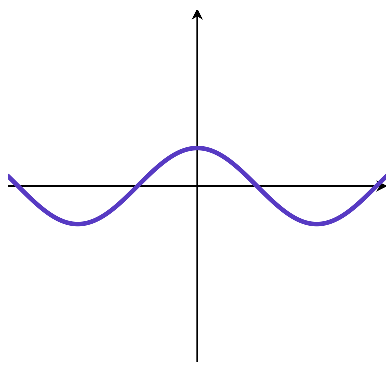
$y = \arcsin(x)$




Domain: $-1 \leq x \leq 1$ Range: $-\pi/2 \leq y \leq \pi/2$

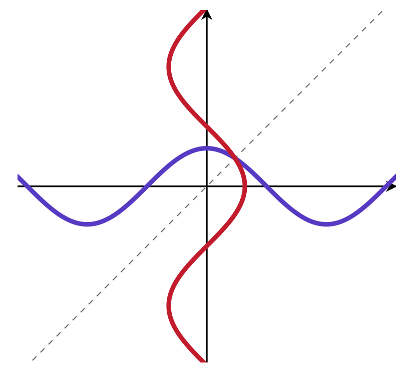
## Domain/range

$y = \cos(x)$



## Domain/range

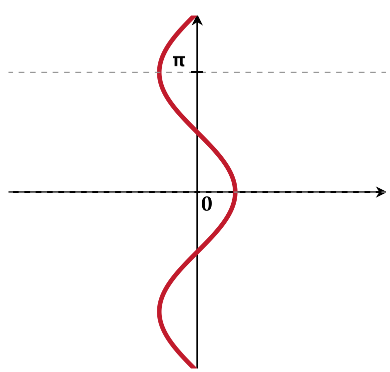
$y = \cos(x)$
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$

## Domain/range

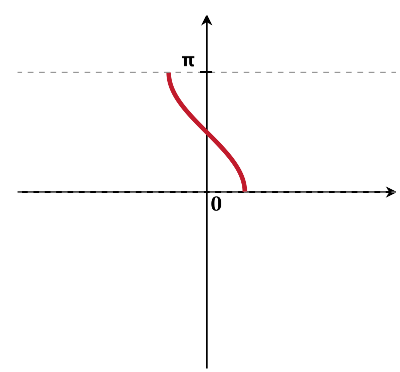
$y = \arccos(x)$




Domain: $-1 \leq x \leq 1$

## Domain/range

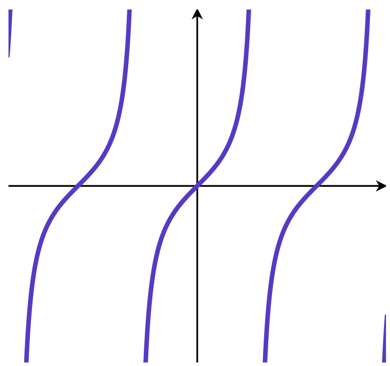
$y = \arccos(x)$




Domain: $-1 \leq x \leq 1$ Range: $0 \leq y \leq \pi$

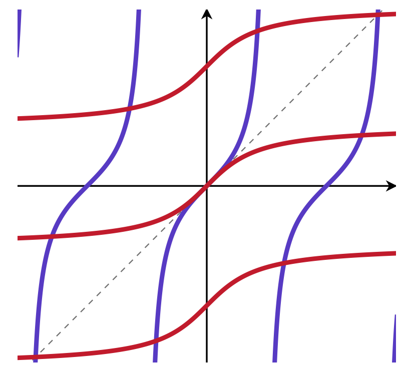
## Domain/range

$y = \tan(x)$



## Domain/range

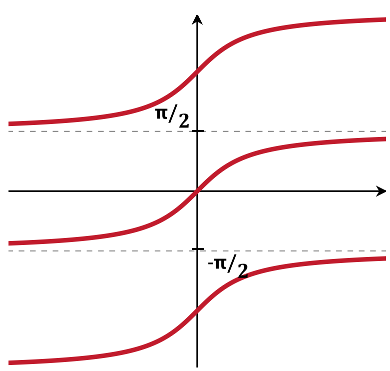
$y = \tan(x)$
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

## Domain/range

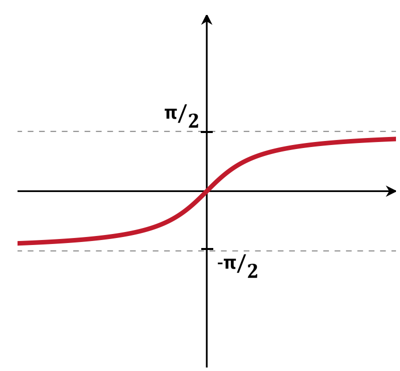
$y = \arctan(x)$




Domain: $-\infty \leq x \leq \infty$

## Domain/range

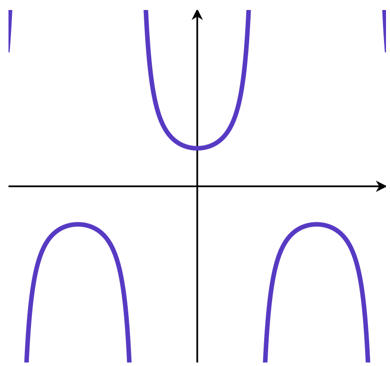
$y = \arctan(x)$




Domain: $-\infty \leq x \leq \infty$ Range: $-\pi/2 < y < \pi/2$

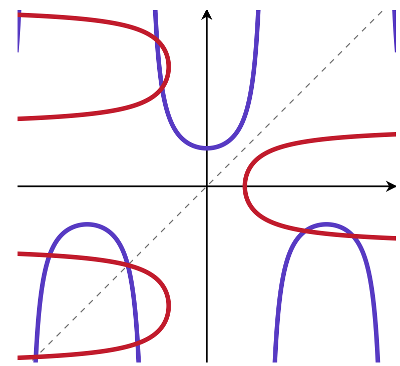
## Domain/range

$y = \sec(x)$



## Domain/range

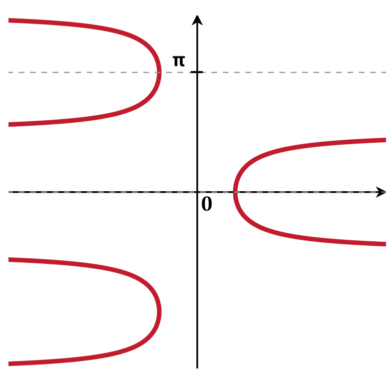
$y = \sec(x)$
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

## Domain/range

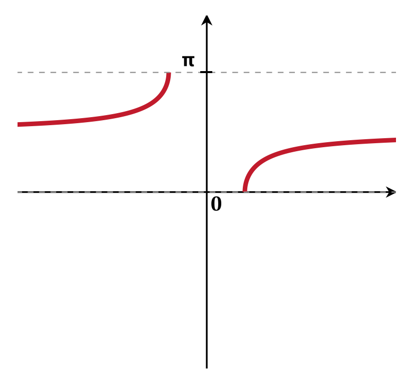
$y = \text{arcsec}(x)$




Domain: $x \leq -1$ and $1 \leq x$

## Domain/range

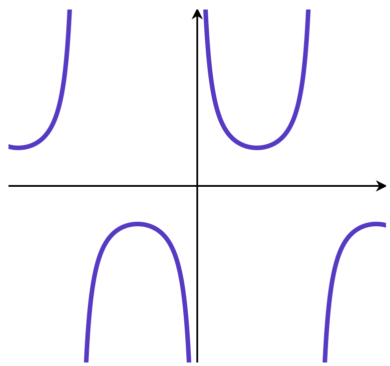
$y = \text{arcsec}(x)$




Domain: $x \leq -1$ and $1 \leq x$ Range: $0 \leq y \leq \pi$

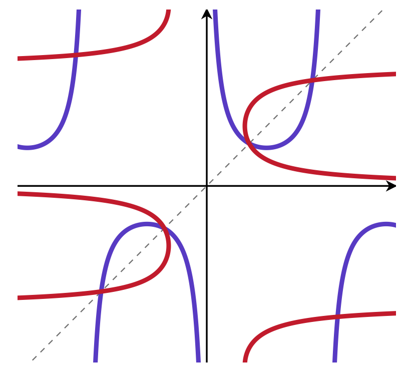
## Domain/range

$y = \csc(x)$



## Domain/range

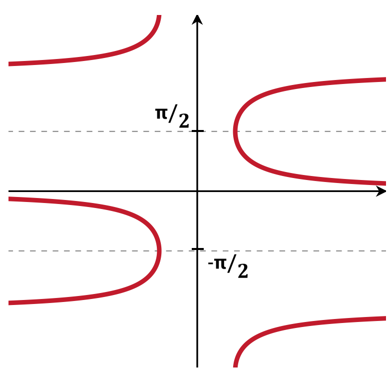
$y = \csc(x)$
$y = \text{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$

## Domain/range

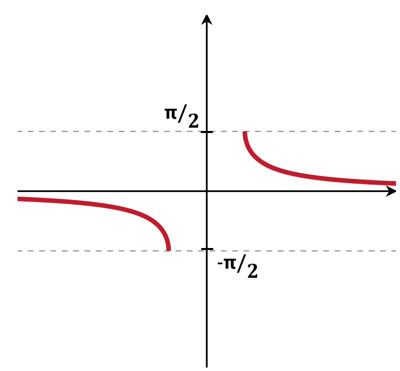
$y = \text{arccsc}(x)$




Domain: $x \leq -1$ and $1 \leq x$

## Domain/range

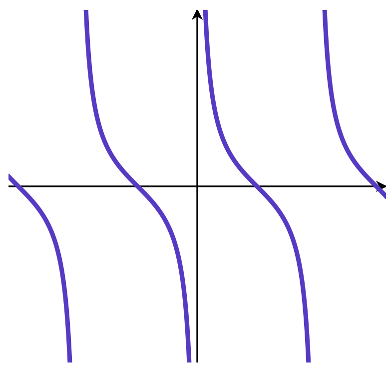
$y = \text{arccsc}(x)$




Domain: $x \leq -1$ and $1 \leq x$ Range: $-\pi/2 \leq y \leq \pi/2$

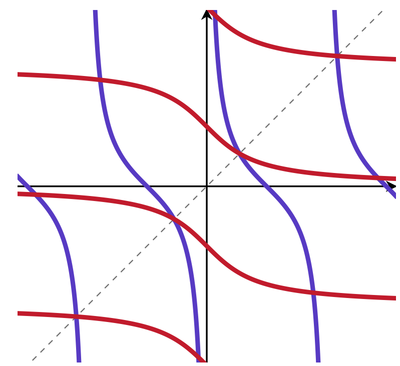
## Domain/range

$y = \cot(x)$



## Domain/range

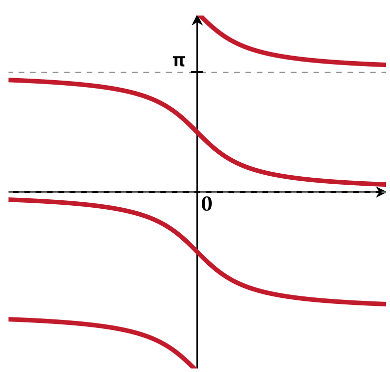
$y = \cot(x)$
$y = \text{arccot}(x)$



Domain: $-\infty \le x \le \infty$

## Domain/range

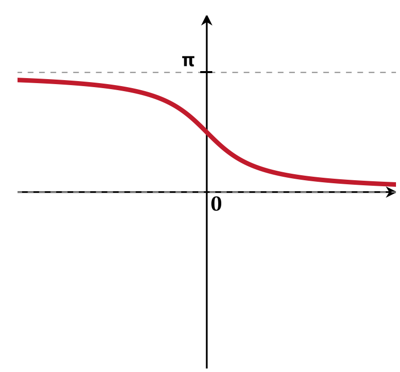
$y = \text{arccot}(x)$




Domain: $-\infty \le x \le \infty$

## Domain/range

$y = \text{arccot}(x)$




Domain: $-\infty \le x \le \infty$ Range: $0 < y < \pi$

# Back to Derivatives

Use implicit differentiation to calculate the derivatives of

1. $\arcsin(x)$
2. $\arctan(x)$

Use the rule

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to check your answers, and then to calculate the derivatives of the other inverse trig functions:

1. $\frac{d}{dx} \arccos(x)$
2. $\frac{d}{dx} \text{arcsec}(x)$
3. $\frac{d}{dx} \text{arccsc}(x)$
4. $\frac{d}{dx} \text{arccot}(x)$

Recall:

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\tan(x)$ | $\sec^2(x)$ |
| $\sec(x)$ | $\sec(x)\tan(x)$ |
| $\csc(x)$ | $-\csc(x)\cot(x)$ |
| $\cot(x)$ | $-\csc^2(x)$ |

## Using implicit differentiation to calculate $\frac{d}{dx}\arcsin(x)$

If $y = \arcsin(x)$ then $x = \sin(y)$.

Take $\frac{d}{dx}$ of both sides of $x = \sin(y)$:

Left hand side: $$\frac{d}{dx}x = 1$$

Right hand side: $$\frac{d}{dx}\sin(y) = \cos(y)*\frac{dy}{dx} = \cos(\arcsin(x))*\frac{dy}{dx}$$

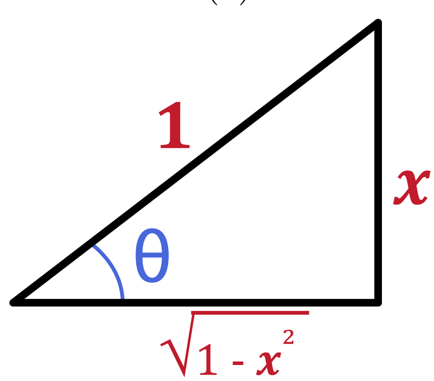
So

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))}.}$$

## Simplifying $\cos(\arcsin(x))$

Call $\arcsin(x) = \theta$.

$\sin(\theta) = x$




So $$\cos(\,\arcsin(x)) = \sqrt{1 - x^2}$$

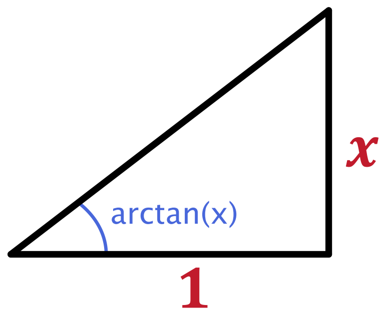
So $$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - x^2}}.$$

## Calculating $\frac{d}{dx}\arctan(x)$.

We found that

$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \left(\frac{1}{\sec(\arctan(x))}\right)^2$$

Simplify this expression using




$$\frac{dy}{dx} = \left(\frac{1}{\sec(\arctan(x))}\right)^2 = \frac{1}{1+x^2}$$

To simplify the rest, use the triangles