INVERSE FUNCTIONS DERIVATIVES

Recall the steps for computing $\frac{dy}{dx}$ implicitly:
(1) Take $\frac{d}{dx}$ of both sides, treating y like a function.
(2) Expand, add, subtract to get the $\frac{dy}{dx}$ terms on one side and everything else on the other.
(3) Factor out $\frac{dy}{dx}$ and divide both sides by its coefficient.

Warmup: Use implicit differentiation to compute $\frac{dy}{dx}$ for the following functions:

1. $e^y = xy$

2. $\cos(y) = x + y$

Inverse functions. Let $f(x)$ be a function. Recall $f^{-1}(x)$ is the *inverse* function of $f(x)$, meaning

$$y = f^{-1}(x) \qquad \text{if and only if} \qquad x = f(y).$$

You can also think of $f^{-1}(x)$ as "undoing" $f(x)$, i.e.

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

For example, $\ln(x)$ is the inverse of e^x, since

$$y = \ln(x) \qquad \text{if and only if} \qquad x = e^y.$$

So

$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x \qquad \text{for } x > 0.$$

Problem 1: Using a calculator or computer, fill out the following table.

x	0	1	2	10	-1	-2	-10
e^x							
$\ln(x)$							
$\ln(e^x)$							
$e^{\ln(x)}$							

What do you notice when x is negative? Explain the pattern.

First: Rewrite

$$y = \ln x \quad \text{as} \quad e^y = x.$$

Next: Take a derivative of both sides of $e^y = x$:

$$\frac{d}{dx}e^y = \boxed{} \qquad \frac{d}{dx}x = \boxed{}.$$

Next: Set your answers equal to each other and solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \boxed{}$$

But there's still a problem! We asked "what is the derivative of $\ln(x)$?" and got back and answer with y in it! To fix this problem, **plug in $y = e^x$ and simplify**. Finally, you should then get

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}.}$$

Problem 2: Compare the graphs of $1/x$ and $\ln(x)$ and check that it makes sense that $\frac{d}{dx}\ln(x) = \frac{1}{x}$.
Problem 3: Using your new identity, $\frac{d}{dx}\ln(x) = \frac{1}{x}$, compute the following.

(1) $\frac{d}{dx}\ln x^2$

(2) $\frac{d}{dx}\ln(\sin(x^2))$

(3) $\frac{d}{dx}\log_3(x)$
[hint: $\log_a x = \frac{\ln x}{\ln a}$]

(4) $\frac{d}{dx}\ln(f(x))$, where $f(x)$ is a differentiable function.

Computing derivatives of inverses in general. In the case where $y = \ln(x)$, we used the fact that $\ln(x) = f^{-1}(x)$, where $f(x) = e^x$, and got

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}.$$

In general, calculating $\frac{d}{dx}f^{-1}(x)$:
First: Rewrite $y = f^{-1}(x)$ as $f(y) = x$.

Next: Use implicit differentiation to compute $\frac{dy}{dx}$:

$$f'(y) * \frac{dy}{dx} = 1 \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{f'(y)}.$$

Finally: Plug back in $y = f^{-1}(x)$ to get an answer in terms of just x:

$$\boxed{\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}.}$$

Problem 4: Check that this identity makes sense by applying it to the following examples.

(1) Compute $\frac{d}{dx}\sqrt{x}$ by noting that
$$\text{if } f(x) = x^2 \text{ then } f^{-1}(x) = \sqrt{x}.$$

(2) Compute $\frac{d}{dx}\sqrt[3]{x}$ by noting that
$$\text{if } f(x) = x^3 \text{ then } f^{-1}(x) = \sqrt[3]{x}.$$

(3) Compute $\frac{d}{dx}\ln(x)$ by noting that
$$\text{if } f(x) = e^x \text{ then } f^{-1}(x) = \ln(x).$$

Inverse trig functions. Recall that we have two notations for inverse trig functions:

$f(x)$	$f^{-1}(x)$
$\sin(x)$	$\sin^{-1}(x) = \arcsin(x)$
$\cos(x)$	$\cos^{-1}(x) = \arccos(x)$
$\tan(x)$	$\tan^{-1}(x) = \arctan(x)$
$\sec(x)$	$\sec^{-1}(x) = \operatorname{arcsec}(x)$
$\csc(x)$	$\csc^{-1}(x) = \operatorname{arccsc}(x)$
$\cot(x)$	$\cot^{-1}(x) = \operatorname{arccot}(x)$

Review the notes for the graphs of these functions.
Also, recall the derivatives of our favorite trig functions:

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$

To compute the derivative of $\arcsin(x)$:

First: Rewrite $y = \arcsin(x)$ as $\sin(y) = x$.

Next: Use implicit differentiation to compute $\frac{dy}{dx}$:

$$\cos(x) * \frac{dy}{dx} = 1 \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{\cos(y)}.$$

Finally: Plug back in $y = \arcsin(x)$ to get an answer in terms of just x:

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x)).}}$$

Problem 5: Use implicit differentiation to calculate the derivative of $\arctan(x)$.
Problem 6: Use the rule

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to check the answers to $\frac{d}{dx}\arcsin(x)$ and $\frac{d}{dx}\arctan(x)$.

Problem 7: Use the rule

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to calculate the derivatives of the other inverse trig functions:

(1) $\frac{d}{dx}\arccos(x)$

(2) $\frac{d}{dx}\text{arcsec}(x)$

(3) $\frac{d}{dx}\text{arccsc}(x)$

(4) $\frac{d}{dx}\text{arccot}(x)$

We'll go over how to simplify these in class on Tuesday.

Solutions to warmup:

(1) $e^y = xy$:

Take $\frac{d}{dx}$ of both sides to find $e^y\frac{dy}{dx} = x\frac{dy}{dx} + y$.

So

$$y = e^y\frac{dy}{dx} - x\frac{dy}{dx} = \frac{dy}{dx}(e^y - x), \quad \text{implying} \quad \boxed{\frac{dy}{dx} = \frac{y}{e^y - x}}$$

(2) $\cos(y) = x + y$:

Take $\frac{d}{dx}$ as before: $-\sin(y)\frac{dy}{dx} = 1 + \frac{dy}{dx}$. So

$$\frac{dy}{dx}(\sin(y) + 1) = -1, \quad \text{and so} \boxed{\frac{dy}{dx} = \frac{-1}{\sin(y) + 1}}.$$

Answers

Answers to Problem 1: Using a calculator or computer, fill out the following table.

x	0	1	2	10	-1	-2	-10
e^x	1	$2.718\ldots$	$7.389\ldots$	$22026.318\ldots$	$0.368\ldots$	$0.135\ldots$	$0.000045\ldots$
$\ln(x)$	Undef	0	$0.693\ldots$	$2.303\ldots$	Undef	Undef	Undef
$\ln(e^x)$	0	1	2	10	-1	-2	-10
$e^{\ln(x)}$	Undef	1	2	10	Undef	Undef	Undef

When x is positive, we have $\ln(e^x) = x$ and $e^{\ln(x)} = x$. But when x is non-positive (0 or negative), then $\ln(e^x) = x$, but $e^{\ln(x)}$ is undefined.

Answers to Problem 3:

(1) $\frac{d}{dx} \ln x^2 = \frac{2x}{x^2} = \frac{2}{x}$

(2) $\frac{d}{dx} \ln(\sin(x^2)) = \frac{2x\cos(x^2)}{\sin(x^2)}$

(3) $\frac{d}{dx} \log_3(x) = \frac{1}{x\ln(3)}$

(4) $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$

Answers to Problem 4:

(1) Compute $\frac{d}{dx}\sqrt{x}$ by noting that

$$\text{if } f(x) = x^2 \text{ then } f^{-1}(x) = \sqrt{x}.$$

Answer. If $f^{-1}(x) = \sqrt{x}$, then $f(x) = x^2$, so that $f'(x) = 2x$. Then

$$\frac{d}{dx}\sqrt{x} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2\sqrt{x}}.$$

Compare this to

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}. \quad \checkmark$$

(2) Compute $\frac{d}{dx}\sqrt[3]{x}$ by noting that

$$\text{if } f(x) = x^3 \text{ then } f^{-1}(x) = \sqrt[3]{x}.$$

Answer. If $f^{-1}(x) = \sqrt[3]{x}$, then $f(x) = x^3$, so that $f'(x) = 3x^2$. Then

$$\frac{d}{dx}\sqrt[3]{x} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3(\sqrt[3]{x})^2} = \frac{1}{3x^{2/3}}.$$

Compare this to

$$\frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}. \quad \checkmark$$

(3) Compute $\frac{d}{dx}\ln(x)$ by noting that

$$\text{if } f(x) = e^x \text{ then } f^{-1}(x) = \ln(x).$$

Answer. If $f^{-1}(x) = \ln(x)$, then $f(x) = e^x$, so that $f'(x) = e^x$. Then

$$\frac{d}{dx}\ln(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}. \quad \checkmark$$

Problem 5: Use implicit differentiation to calculate the derivative of $\arctan(x)$.

Answer. We have $y = \arctan(x)$ if and only if $\tan(y) = x$. Taking $\frac{d}{dx}$ of both sides gives

$$\frac{d}{dx}\tan(y) = \sec^2(y)\frac{dy}{dx} \qquad \text{and} \qquad \frac{d}{dx}x = 1.$$

So

$$\sec^2(y)\frac{dy}{dx} = 1, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{(\sec(y))^2} = \boxed{\frac{1}{(\sec(\tan(x)))^2}}.$$

Problem 6: Use the rule

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to check the answers to $\frac{d}{dx}\arcsin(x)$ and $\frac{d}{dx}\arctan(x)$.

Answer. If $f^{-1}(x) = \arcsin(x)$, then $f(x) = \sin(x)$, so that $f'(x) = \cos(x)$. Then

$$\frac{d}{dx}\arcsin(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin(x))}. \quad \checkmark$$

If $f^{-1}(x) = \arctan(x)$, then $f(x) = \tan(x)$, so that $f'(x) = \sec^2(x)$. Then

$$\frac{d}{dx}\arctan(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\arctan(x))}. \quad \checkmark$$

Problem 7: Use the rule

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to calculate the derivatives of the other inverse trig functions:

(1) $\frac{d}{dx}\arccos(x)$

Answer. If $f^{-1}(x) = \arccos(x)$, then $f(x) = \cos(x)$, so that $f'(x) = -\sin(x)$. Then

$$\frac{d}{dx}\arcsin(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\sin(\arccos(x))}.$$

(2) $\frac{d}{dx}\text{arcsec}(x)$

Answer. If $f^{-1}(x) = \text{arcsec}(x)$, then $f(x) = \sec(x)$, so that $f'(x) = \sec(x)\tan(x)$. Then

$$\frac{d}{dx}\text{arcsec}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec(\text{arcsec}(x))\tan(\text{arcsec}(x))}.$$

(3) $\frac{d}{dx}\text{arccsc}(x)$

Answer. If $f^{-1}(x) = \text{arccsc}(x)$, then $f(x) = \csc(x)$, so thar $f'(x) = csc(x)\cot(x)$. Then

$$\frac{d}{dx}\text{arccsc}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\csc(\text{arcsec}(x))\cot(\text{arcsec}(x))}.$$

(4) $\frac{d}{dx}\text{arccot}(x)$

Answer. If $f^{-1}(x) = \text{arccot}(x)$, then $f(x) = \cot(x)$, so that $f'(x) = -\csc^2(x)$. Then

$$\frac{d}{dx}\text{arccot}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\csc^2(\text{arccot}(x))}.$$