Warm-up

Suppose $f(x)$ is a differentiable function. Then in terms of x, $f(x)$ and $f'(x)$, what are

1. $\frac{d}{dx}(xf(x))$
2. $\frac{d}{dx}(f(x) + \sin(x))$
3. $\frac{d}{dx}(f(x)/x)$
4. $\frac{d}{dx}\left(f^2(x)\right)$
5. $\frac{d}{dx}\left(1 + xf^4(x)\right)$
6. $\frac{d}{dx}\left(\cos\left(xe^{f(x)}\right)\right)$

Important reminder:

The notes from class are **not comprehensive**!

To be fully prepared for homework and exams, you must also do the **reading from the book**.

The approximate reading schedule for the entire semester, as well as book problems analogous to those on the homework, are posted on the website.

Implicit functions

An implicit function is something that can we written as

$$F(x, y) = c.$$

So it might not be a function, but it is a curve in the x-y plane.

For example, $x^2 + y^2 = 1$.

Question:

What is the slope of the tangent line to the unit circle at $x = \frac{\sqrt{2}}{2}$?

Using calculus so far...

Solve for y:

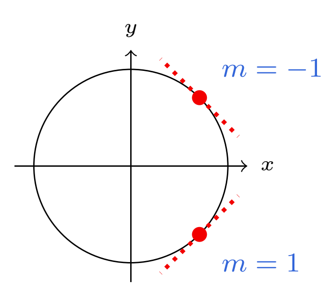
$$y = \pm\sqrt{1 - x^2}$$

Take a derivative:

$$\frac{dy}{dx} = -\frac{\pm 2x}{2\sqrt{1-x^2}}$$

Plug in the point:

$$\left.\frac{dy}{dx}\right|_{x=\frac{\sqrt{2}}{2}} = -\frac{\pm\sqrt{2}}{2\sqrt{1-1/2}} = \mp 1$$



But which is which??

...we can do better.

Implicit differentiation

Instead, think of y as a function of x: let $y = f(x)$.
Then rewrite

$$x^2 + y^2 = 1 \quad \text{as} \quad x^2 + f^2(x) = 1.$$

(1) Take a derivative of both sides independently:

$$\frac{d}{dx}(x^2 + f^2(x)) = 2x + 2f(x) * f'(x) = 2x + 2y\frac{dy}{dx}$$

$$\frac{d}{dx}1 = 0$$

(2) So $\boxed{2x + 2y\frac{dy}{dx} = 0}$

(3) solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -2x/2y = -x/y.}$$

(4) Plug in points:

x	y	$\frac{dy}{dx}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1

Implicit differentiation

In general, given $F(x, y) = c$, calculate $\frac{dy}{dx}$ in terms of x and y.

Step 1: Take derivative of both sides versus x.
(don't forget the chain rule!)

Step 2: Set the two sides equal.

Step 3: Solve for $\frac{dy}{dx}$.

Step 4: If applicable, plug in points.

For example, calculate $\frac{dy}{dx}$ if $xy^2 = 3$.

(1) $\frac{d}{dx}xy^2 = 1 * y^2 + x * \left(2y * \frac{dy}{dx}\right)$ and $\frac{d}{dx}3 = 0$

(2) So $y^2 + 2xy\frac{dy}{dx} = 0$

(3) and so $\boxed{\frac{dy}{dx} = -y^2/2xy = \frac{-y}{2x}}$

Examples

Calculate $\frac{dy}{dx}$ if...

(a) $xy = 1$

(b) $y + \sin(x) = xy$

(c) $y/x = xy^4 + 1$

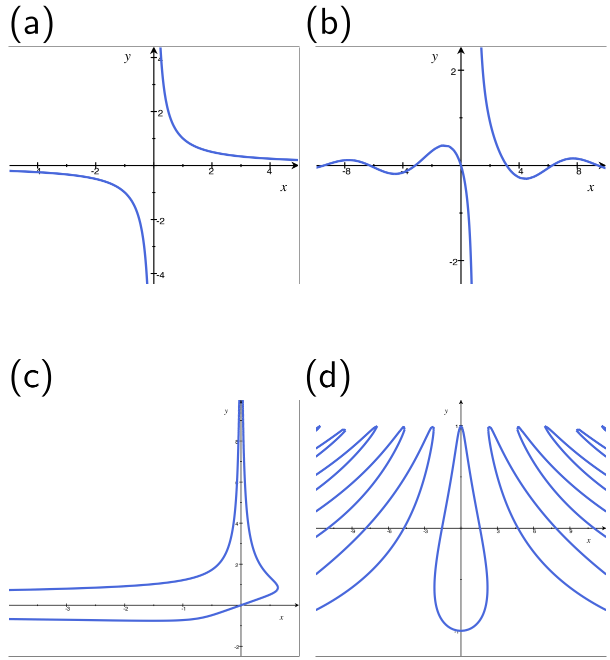
(d) $\cos(xe^y) = y^2$

Then $\frac{dx}{dy}$ for (a) and (b) as well. To do this, now pretend that $x = f(y)$. See a pattern?

In general,

$$\boxed{\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right).}$$

For reference, the graphs look like



Higher derivatives;

Calculate $\frac{d^2y}{dx^2} = y''$ if $xy^2 = 3$.

You can start from either

$$y^2 + 2xyy' = 0 \qquad \text{or} \qquad y' = -y/2x.$$

$$\begin{aligned} y'' = \frac{d}{dx}(-y/2x) &= -\frac{y' * 2x - y * 2}{(2x)^2} \\ &= -\frac{y'x - y}{2x^2} \\ &= -\frac{(-y/2x)x - y}{2x^2} = \boxed{\frac{3y}{4x^2}} \end{aligned}$$

Get your answer back into just x's and y's!

Vertical and horizontal tangent lines

We saw that if $x^2 + y^2 = 1$, then $\frac{dy}{dx} = -x/y$.

Question 1: At which points is the tangent line horizontal?
Solve $0 = \frac{dy}{dx}$:

$$0 = -x/y \quad \text{if } x = 0$$

To get the y-coordinate, plug back in and solve:

$$0^2 + y^2 = 1, \quad \text{so } y = \pm 1.$$

Answer: $\boxed{\text{at } (0,1) \text{ and } (0,-1).}$

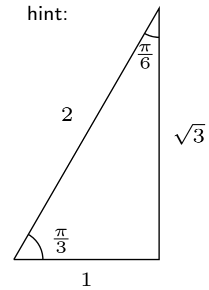
Question 2: At which points is the tangent line vertical?
Solve: where is $\frac{dy}{dx}$ approaching $\pm\infty$? At $y = 0$!
To get the x-coordinate, plug back in and solve:

$$x^2 + 0^2 = 1, \quad \text{so } x = \pm 1.$$

Answer: $\boxed{\text{at } (1,0) \text{ and } (-1,0).}$

Other tangent lines

We saw that if $x^2 + y^2 = 1$, then $\frac{dy}{dx} = -x/y$.
Question 3: Find the equations for all lines which are tangent to the unit circle and have slope $\sqrt{3}$.



Solution: Solve $\sqrt{3} = \frac{dy}{dx} = -x/y$.
This happens when

$$\boxed{\cos(\theta)/\sin(\theta) = x/y = -\sqrt{3}.}$$

So $\theta = -\pi/6$ or $\theta = \pi - \pi/6 = 5\pi/6$.

At $\theta = -\pi/6$, we have $x = \sqrt{3}/2, y = -1/2$, so the tangent line is

$$\boxed{y = \sqrt{3}(x - \sqrt{3}/2) - 1/2}$$

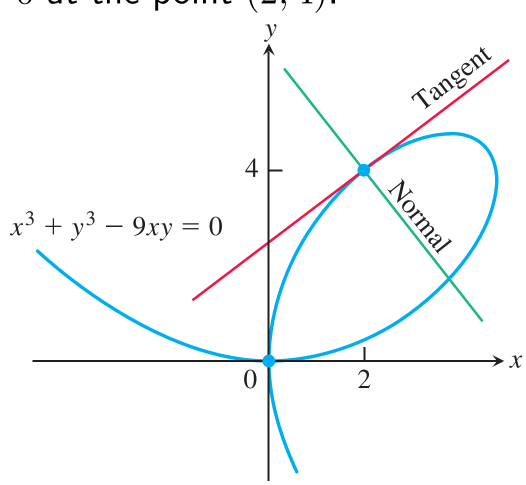
At $\theta = 5\pi/6$, we have $x = -\sqrt{3}/2, y = 1/2$, so the tangent line is

$$\boxed{y = \sqrt{3}(x + \sqrt{3}/2) + 1/2}$$

Tangent and normal lines

The line normal to a curve $F(x, y) = c$ at a point (x_0, y_0) is the line that goes through (x_0, y_0) and is perpendicular to the tangent line. *(Recall if $\ell_1 \perp \ell_2$ are perpendicular lines with slopes m_1 and m_2, respectively, then $m_2 = -1/m_1$.)*

Example: Compute the lines tangent and normal to the curve $x^3 + y^3 - 9xy = 0$ at the point $(2, 4)$.



$$3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0, \quad \text{so } \frac{dy}{dx} = (-3x^2 + 9y)/(3y^2 - 9x)$$

Examples

Calculate the lines normal to each curve at the given point. . .

(a) $xy = 1$
at $(-2, -1/2)$

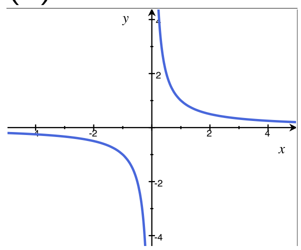
(b) $y + \sin(x) = xy$
at $(0, 0)$

(c) $y/x = xy^4 + 1$
at $(0, 0)$

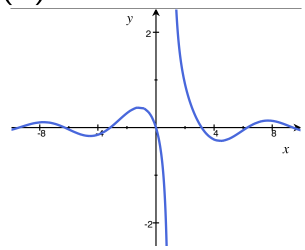
(d) $\cos(xe^y) = y^2$
at $(0, -1)$

For reference, the graphs look like

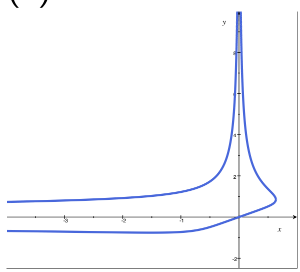
(a)



(b)



(c)



(d)