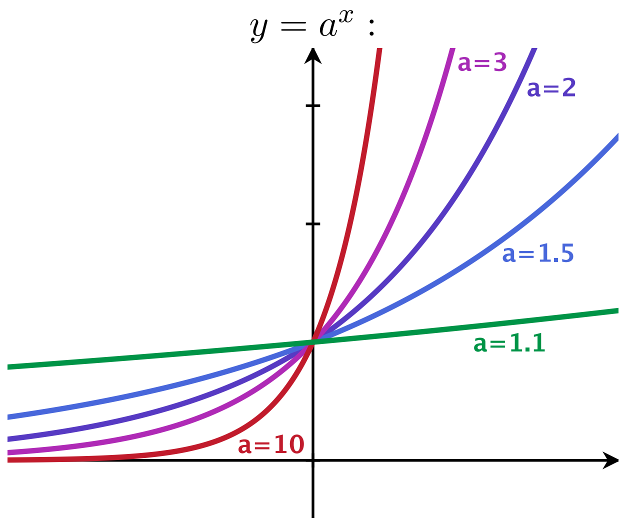
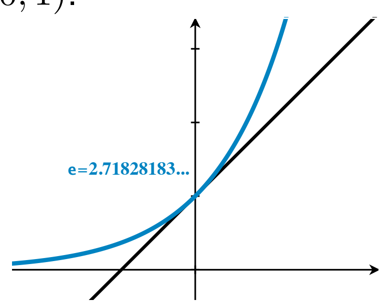
Recall: Our favorite exponential function

Look at how the curve $y = a^x$ is increasing through the point $(0, 1)$:



Derivative of exponential functions

We *defined* e as the number such that the curve $y = e^x$ has slope $m = 1$ at the point $(0, 1)$.



This means

$$1 = \frac{d}{dx} e^x \Big|_{x=0} = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

So we may take for granted that

$$\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}.$$

Derivative of exponential functions

$$\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

Now, let's compute $\frac{d}{dx}e^x$:

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \qquad \text{(since } x \text{ is constant in the limit } h \to 0) \\
&= e^x \cdot 1 = e^x.
\end{aligned}$$

So

$$\boxed{\frac{d}{dx}e^x = e^x}.$$

Derivative of exponential functions

$$\boxed{\frac{d}{dx}e^x = e^x}$$

What about $\frac{d}{dx}a^x$ for other numbers a?
Recall that $\ln(x)$ is the inverse function of e^x, so that

$$e^{\ln(y)} = y.$$

Therefore,

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)}, \qquad \text{since } \ln(a^x) = x\ln(a).$$

Recall chain rule: $\boxed{\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)}$.

Here, we can write

$$f(x) = e^x \text{ and } g(x) = \ln(a)x \qquad \text{so that } f(g(x)) = e^{\ln(a)x} = a^x.$$

Thus, since

$$f'(x) = e^x, \quad g'(x) = \ln(a), \quad \text{and} \quad f'(g(x)) = e^{\ln(a)x} = a^x,$$

we have

$$\boxed{\frac{d}{dx}a^x = f'(g(x)) \cdot g'(x) = a^x \cdot \ln(a)}.$$

Derivative of exponential functions

$$\boxed{\frac{d}{dx}e^x = e^x} \qquad \boxed{\frac{d}{dx}a^x = a^x \cdot \ln(a)}$$

Note: $\ln(e) = 1$, so these rules agree when $a = e$. ✓

You try: Compute the derivatives of the following equations.

1. $f(x) = 2^x$
2. $f(x) = 3^x$
3. $f(x) = (1/5)^x$
4. $f(x) = (2e)^x$
5. $f(x) = e^{2x+1}$
6. $f(x) = 3^{4x^2-5x+e^x}$

Some applications of derivatives

Let x_0 be a real number. The **instantaneous rate of change** of $f(x)$ with respect to x, at x_0, is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example. The area of a circle of radius r is $A = \pi r^2$. Suppose the radius of a circle is varying—how do the area vary with respect to the change in radius?

Variable: r Function: $A = A(r)$

Answer:

$$\frac{d}{dr}A = \frac{d}{dr}(\pi r^2) = 2\pi r.$$

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Question: What is the rate of change of the area with respect to the radius when the radius is 5 m?

Answer:

$$\frac{d}{dr}A\Big|_{r=5} = 2\pi r\Big|_{r=5} = 10\pi.$$

Question: Suppose the radius is changing at a rate of 7 m/s. How fast is the area changing when the radius is $5m$?

Variable: t Function: $A = A(r(t)) = \pi(r(t))^2$

Some applications of derivatives

Question: Suppose the radius is changing at a rate of 7 m/s. How fast is the area changing when the radius is 5 m?

Variable: t Function: $A = A(r(t)) = \pi(r(t))^2$

Answer:

$$\frac{d}{dt}A = \frac{d}{dt}A(r(t)) = \underbrace{\frac{dA}{dr} \cdot \frac{dr}{dt}}_{\text{Use Leibnitz notation!}} = (2\pi r(t))r'(t).$$

We don't know what time t_0 we care about, but we do know that *at that (mystery) time*,

$$r(t_0) = 5 \qquad \text{and} \qquad r'(t_0) = 7.$$

So

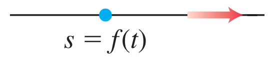
$$\left.\frac{d}{dt}A\right|_{t=t_0} = 2\pi \cdot r(t_0) \cdot r'(t_0) = 2\pi \cdot 5 \cdot 7 = \boxed{70\pi}.$$

Look up "related rates" online.

Motion Along a Line

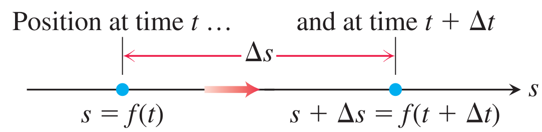
Let an object move (in time) back and forth along a line according to the function

$$s = f(t).$$



The **displacement** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t).$$



The **average velocity** of the object over that time interval is

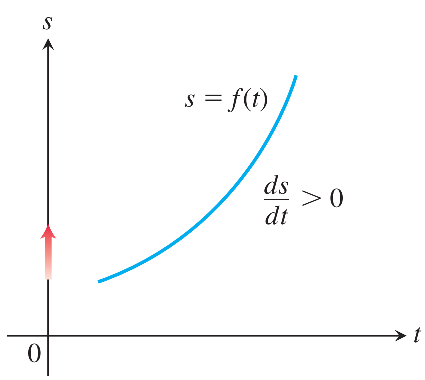
$$v_{\text{av}} = \frac{\text{displacement}}{\text{time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

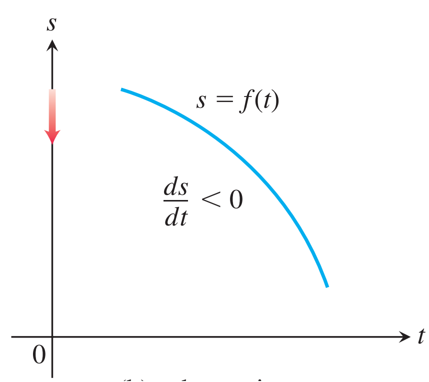
(Think: Δt is just like h from before!)
The **velocity** of the object as a function of time is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Speed versus velocity

If the object is moving *forward*, the velocity is positive.





Similarly, if the object is moving *backwards*, the velocity is negative.

The **speed** of the object is the absolute value of velocity,

$$\text{speed} = |v(t)| = \left| \frac{ds}{dt} \right|.$$

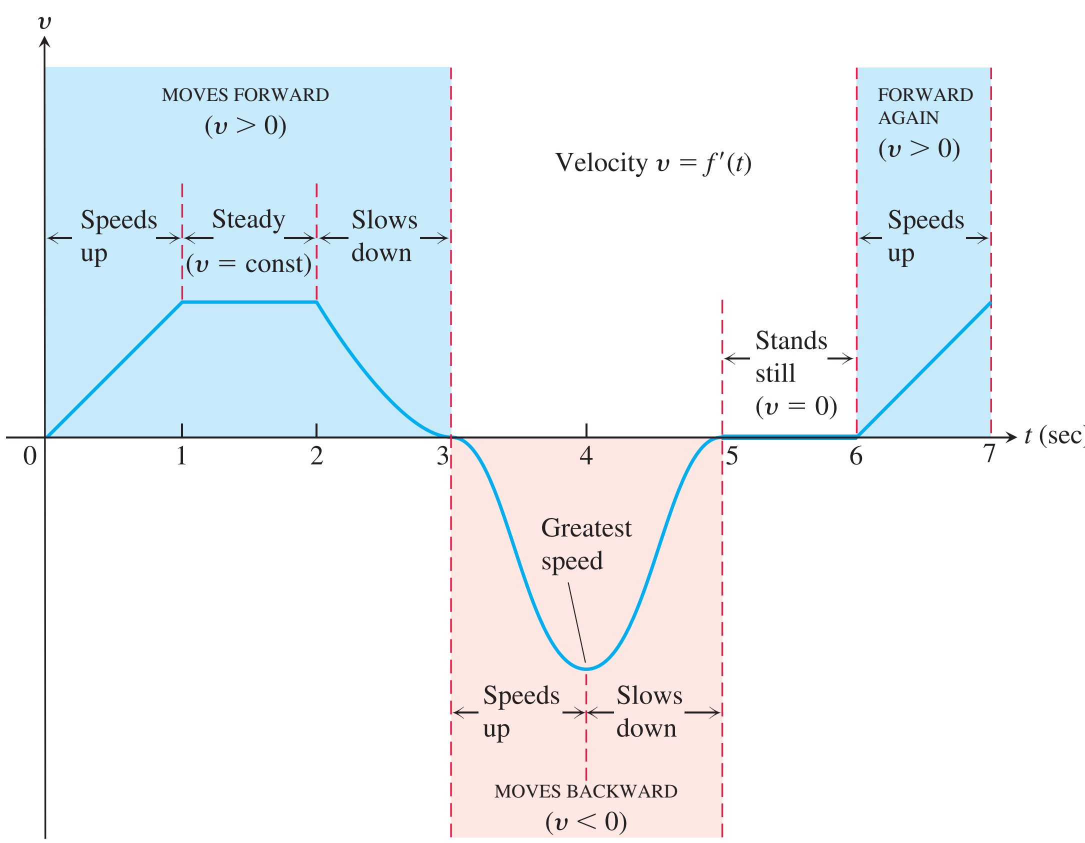
Change in velocity

Acceleration $a(t)$ is the change in velocity over time. Namely, it is the derivative of velocity with respect to time:

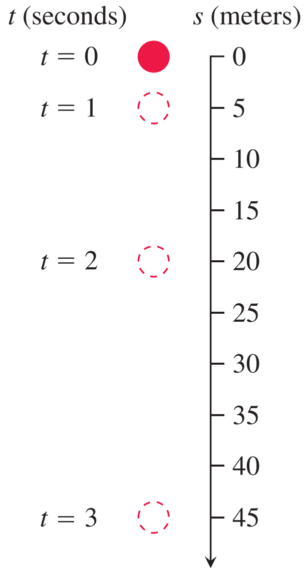
$$a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}s(t).$$

Jerk is the change in acceleration over time, i.e. the derivative of acceleration with respect to time:

$$j(t) = \frac{d}{dt}a(t) = \frac{d^2}{dt^2}v(t) = \frac{d^3}{dt^3}s(t).$$



Acceleration under gravity



Near the surface of Earth all bodies <u>fall</u> with the same constant acceleration:

$$s = \frac{1}{2}gt^2, \quad \text{where} \quad g = 9.8\text{m/s}^2.$$

(Frame of reference: positive is down.)

Check:

$$v = \frac{d}{dt}s = \frac{d}{dt}\left(\frac{1}{2}gt^2\right) = \frac{1}{2}\cdot g\cdot 2t = gt.$$

$$a = \frac{d}{dt}v = \frac{d}{dt}gt = g. \quad \checkmark$$

Example: We drop a ball from a very high tower. How far has it fallen after 10 seconds? How fast is it going at that point?
Ans.

$$s(10) = \frac{1}{2}\cdot g\cdot(10)^2 = 980/2 \text{ meters}, \quad v(10) = g(t) = 98\text{m/s}.$$

Acceleration under gravity

Near the surface of Earth, the vertical trajectory of a body is given by

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0,$$

where

$$g = -9.8\text{m/s}^2,$$

$$v_0 = \text{initial velocity (m/s), and}$$

$$s_0 = \text{initial position (m).}$$

(Frame of reference: positive is up.)

Example. A cannonball is shot up in the air from 1 meter above the ground at an initial velocity of 400 m/s.
(1) What is the maximum height that the ball reaches?

This happens when $v = 0$.

(2) When does the ball hit the ground?

This happens when $s = 0$.

Use $\boxed{s(t) = \frac{1}{2}(-9.8)t^2 + 400t + 1}$.

Rates of change in economics

Let's consider the cost of production $c(x)$ as a function of x, the number of units produced. (The first thing is expensive to make; manufacturing in bulk can be more efficient.) If you're already producing x things, the average cost of producing h more units is

$$\frac{c(x+h)-c(x)}{h} = \frac{\text{extra cost of producing } h \text{ more things}}{\text{number of extra things}}.$$

Then the **marginal cost of production** is

$$\text{marginal cost} = \lim_{h\to 0} \frac{c(x+h)-c(x)}{h} = \frac{d}{dx}c(x).$$

(Sometimes, h isn't relatively small compared to x, in which case we can only really look as small as $h=1$.)

Read examples 5 and 6 in the book.

See also "sensitivity to change", e.g. with genetic data. (Example 7)

Review

- Functions, basic graphs, graph transformations
- Domains and ranges
- Trig functions and identities, inverse trig functions
- Exponential functions, logarithms, and identities
- Limits
  - one- and two-sided
  - when are they defined
  - computing them
- Asymptotes
- Continuity
- Average rate of change
- Limit definition of derivatives
  - polynomials, roots, reciprocals
- Basic derivative rules
  - powers, scalars, sums, products, compositions

Functions, basic graphs, graph transformations

Know basic graphs of

$$mx + b, \quad x^2, \quad x^3, \quad x^4.$$
$$1/x, \quad 1/x^2, \quad \sqrt{x}, \quad \sqrt[3]{x}.$$

If you know the graph of $y = f(x)$, also know the graphs of

$$f(x + c), \quad f(cx), \quad cf(x), \quad f(x) + c, \quad 1/f(x).$$

Also know how graph transformations affect domain and range.

Trig functions and identities, inverse trig functions

- Graphs of $\sin(x)$ and $\cos(x)$
- How to use the unit circle
- Special values
- Angle addition formulas
- How to compute $\tan(x), \csc(x), \sec(x), \cot(x)$ and their graphs

Exponential functions, logarithms, and identities

- Graphs of a^x for $a > 0$ and for $a < 0$
- What is e?
- Identities like $a^{x+y} = a^x a^y$, etc.
- Graphs of $\log_a(x)$. What is $\ln(x)$?
- Identities like $\ln(xy) = \ln(x) + \ln(y)$.
- Exponential growth.

Limits and continuity

- One sided limits, from the left or right
- Two-sided limits
- Limits at $\pm\infty$
- Infinite limits
- Computing standard limits
- Graph asymptotes (vertical, horizontal, skew).
- Definition of continuous, and how to compute where functions are discontinuous.
  (Difference between the limit existing and a function being continuous.)

Rates of change

- Average rate of change
  $$\frac{f(x+h)-f(x)}{h}$$
- Limit definition of derivative
  $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$
- Computing derivatives of functions like $mx+b, x^2, x^3, \sqrt{x}, 1/x, 1/x^2$ using the limit definition.
- Basic derivative rules: powers, scalars, sums, products, compositions