

Warmup: Use the limit definition of the derivative to calculate the following derivatives.

1. $\frac{d}{dx}(5x + 2)$

2. $\frac{d}{dx}(3x - 1)$

3. $\frac{d}{dx}[(5x + 2) + (3x - 1)]$

4. $\frac{d}{dx}[(5x + 2)(3x - 1)]$

5. $\frac{d}{dx}15x^2$

6. $\frac{d}{dx}(15x^2 + x - 2)$

Remember the power rule says $\frac{d}{dx}x^a = ax^{a-1}$.

Based on your calculations above, which of the following statements seem to be true and which seem to be false?

- (a) If you multiply a function $f(x)$ by a number c and then take a derivative, you get the same thing as taking the derivative $f'(x)$ and then multiplying by c . (try comparing 5 to the power rule)
- (b) If you add two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then adding those together. (try comparing 1-3, and then 6 to the power rule)
- (c) If you multiply two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then multiplying those together. (try comparing 1, 2, and 4)

Multiplying by constants: what's going on?

Take another look at $f(x) = 15x^2$. Before, we just expanded and canceled, and were surprised to find something nice happened:

$$\begin{aligned}\frac{d}{dx} [15x^2] &= \lim_{h \rightarrow 0} \frac{15(x+h)^2 - 15x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{15x^2 + 30xh + 15h^2 - 15x^2}{h} \\ &= \lim_{h \rightarrow 0} 30x + \underbrace{15h}_{\substack{\downarrow \\ 0}} = \boxed{30x}\end{aligned}$$

Let's try again, only pay closer attention to that 15:

$$\begin{aligned}\frac{d}{dx} 15x^2 &= \lim_{h \rightarrow 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h \rightarrow 0} 15 * \frac{(x+h)^2 - x^2}{h} \\ &= 15 * \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= 15 * \frac{d}{dx} x^2\end{aligned}$$

one of our limit rules!

Let's try again, only pay closer attention to that 15:

$$\begin{aligned}\frac{d}{dx} 15x^2 &= \lim_{h \rightarrow 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h \rightarrow 0} 15 * \frac{(x+h)^2 - x^2}{h} \\ &= 15 * \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= 15 * \frac{d}{dx} x^2\end{aligned}$$

one of our limit rules!

But now suppose you have any differentiable function $f(x)$ and a number c . [Think: $f(x) = x^2$ and $c = 15$]. Then *in general*

$$\begin{aligned}\underline{\underline{\frac{d}{dx} (c * f(x))}} &= \lim_{h \rightarrow 0} \frac{c * f(x+h) - c * f(x)}{h} \\ &= \lim_{h \rightarrow 0} c * \frac{(f(x+h) - f(x))}{h} \\ \text{limit rule} &= c * \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))}{h} = \underline{\underline{c * \frac{d}{dx} f(x)}}$$

Multiplying by constants

Theorem (Scalars)

If $y = f(x)$ is a differentiable function and c is a constant, then

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x).$$

Example

Since $\frac{d}{dx}x^2 = 2x$, we have $\frac{d}{dx}15x^2 = 15 \cdot (2x) = 30x$.

Taking sums: what's going on?

Take another look at $f(x) = (5x + 2) + (3x - 1)$. Before, we just simplified first, and were surprised:

$$\frac{d}{dx} [(5x+2) + (3x-1)] = \frac{d}{dx} [8x + 1] = \lim_{h \rightarrow 0} \frac{8(x+h) + 1 - (8x+1)}{h} = \lim_{h \rightarrow 0} \frac{8h}{h} = \boxed{8}$$

simplify first

Let's try again, only pay closer attention to either part of the sum:

$$\begin{aligned} \frac{d}{dx} ((5x+2) + (3x-1)) &= \lim_{h \rightarrow 0} \frac{(5(x+h)+2) + (3(x+h)-1) - [5x+2 + 3x-1]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{(5(x+h)+2) - (5x+2)}{h} + \frac{(3(x+h)-1) - (3x-1)}{h} \right] \\ &\stackrel{\text{because } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}}{=} \lim_{h \rightarrow 0} \frac{5(x+h)+2 - (5x+2)}{h} + \lim_{h \rightarrow 0} \frac{3(x+h)-1 - (3x-1)}{h} \\ &\stackrel{\text{limit rule!}}{=} \lim_{h \rightarrow 0} \frac{5(x+h)+2 - (5x+2)}{h} + \lim_{h \rightarrow 0} \frac{3(x+h)-1 - (3x-1)}{h} \\ &= \frac{d}{dx} (5x+2) + \frac{d}{dx} (3x-1) \end{aligned}$$

Now, suppose you have any differentiable functions $f(x)$ and $g(x)$ [Think: $f(x) = 5x + 2$ and $g(x) = 3x - 1$]. Then *in general*

$$\begin{aligned} \frac{d}{dx} [f(x) + g(x)] &= \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \quad (\text{because } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}) \\ &\stackrel{\text{limit rule!}}{=} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \end{aligned}$$

Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x)$$

Example

Use the three rules we have so far

$$\frac{d}{dx}x^a = ax^{a-1}, \quad \frac{d}{dx}(c \cdot f(x)) = c \cdot \left(\frac{d}{dx}f(x)\right),$$

$$\text{and } \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

to calculate the derivatives:

1. $\frac{d}{dx}(x^3 - 7x^2 + 6x^{-15})$

2. $\frac{d}{dx} \left(\sqrt{x} + 100 \sqrt[17]{x^3} - \frac{3}{x^{19}} \right)$

[hint: rewrite everything from 2 as powers before you do anything]

Products: What's going on?

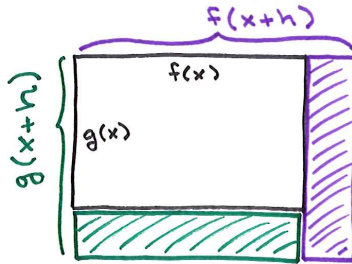
Take another look at $f(x) = (5x + 2) \cdot (3x - 1)$. Before, we just simplified first, and were... not surprised:

$$\begin{aligned} \frac{d}{dx}[(5x+2)(3x-1)] &= \frac{d}{dx}(15x^2 + x - 2) \\ &= \lim_{h \rightarrow 0} \frac{15(x+h)^2 + (x+h) - 2 - (15x^2 + x - 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (15x^2 + \underline{30xh} + \underline{15h^2} + x + \underline{h} - 2 - 15x^2 - x + 2) \\ &= \lim_{h \rightarrow 0} 30x + \underline{15h} + 1 = \boxed{30x + 1} \end{aligned}$$

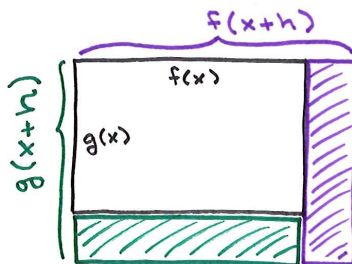
We *didn't* get that the derivative of the products is the product of the derivatives! So what *is* going on here?

To understand how to deal with products, we're going to have to unpack the formula

$$\frac{d}{dx} (f(x) \cdot g(x)) = \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$



$$\begin{aligned} f(x+h) \cdot g(x+h) - f(x) \cdot g(x) &= \underbrace{\text{[green hatched rectangle]}}_{f(x) \cdot (g(x+h) - g(x))} + \underbrace{\text{[purple hatched rectangle]}}_{g(x+h) \cdot (f(x+h) - f(x))} \\ &= f(x) \cdot (g(x+h) - g(x)) \\ &\quad + g(x+h) \cdot (f(x+h) - f(x)) \end{aligned}$$



$$\begin{aligned} f(x+h) \cdot g(x+h) - f(x) \cdot g(x) &= \underbrace{\text{[green hatched rectangle]}}_{f(x) \cdot (g(x+h) - g(x))} + \underbrace{\text{[purple hatched rectangle]}}_{g(x+h) \cdot (f(x+h) - f(x))} \\ &= f(x) \cdot (g(x+h) - g(x)) \\ &\quad + g(x+h) \cdot (f(x+h) - f(x)) \end{aligned}$$

So

$$\begin{aligned} \frac{d}{dx} f(x) \cdot g(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(f(x) \cdot [g(x+h) - g(x)] + g(x+h) \cdot [f(x+h) - f(x)] \right) \end{aligned}$$

So

$$\begin{aligned}\frac{d}{dx} f(x) \cdot g(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(f(x) \cdot [g(x+h) - g(x)] + g(x+h) \cdot [f(x+h) - f(x)] \right) \\ &= \lim_{h \rightarrow 0} \left[f(x) \cdot \left(\frac{g(x+h) - g(x)}{h} \right) + g(x+h) \cdot \left(\frac{f(x+h) - f(x)}{h} \right) \right] \\ &= f(x) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \quad \rightarrow g'(x) \\ &\quad + \left(\lim_{h \rightarrow 0} g(x+h) \right) \cdot \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) \\ &\quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ &\quad \quad \quad g(x) \quad \quad \quad f'(x) \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x)\end{aligned}$$

Theorem (Products)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

Example: Calculate $\frac{d}{dx} ((5x + 2)(3x - 1))$:

$$\begin{aligned}\frac{d}{dx} ((5x + 2)(3x - 1)) &= (5x + 2) \cdot 3 + (3x - 1) \cdot 5 = \boxed{30x + 1} \quad \odot \\ f \uparrow \quad \quad g \uparrow \quad \quad f \cdot g' + g \cdot f'\end{aligned}$$

Last rule: Compositions.

Example: Calculate $\frac{d}{dx}(5x + 2)^{100}$.

If $f(x) = x^{100}$ and $g(x) = 5x + 2$, then $f(g(x)) = (5x + 2)^{100}$.

So since $f'(x) = 100x^{99}$ and $g'(x) = 5$, if everything were easy in the world, we might hope that

$$\frac{d}{dx}(5x + 2)^{100} = 100(5)^{99}$$

But it's not!!

$$\frac{d}{dx}(5x + 2)^{100} \neq 100(5)^{99}$$

Theorem (Chain rule)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}((f \circ g)(x)) = \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Last rule: Compositions.

Why:

$$\begin{aligned} \frac{d}{dx} f(g(x)) &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \\ &\quad \begin{array}{l} \downarrow \\ \text{how } f \text{ changes versus } \\ g(x) \text{ (instead of} \\ \text{versus } x) \end{array} \quad \begin{array}{l} \downarrow \\ g'(x) \end{array} \\ &= f'(g(x)) \end{aligned}$$

In Leibniz notation:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$$\text{Chain rule: } \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Example

$$\text{Calculate } \frac{d}{dx}(5x + 2)^{100}.$$

Here,

$$f(x) = x^{100} \quad \text{and} \quad g(x) = 5x + 2.$$

So

$$f'(x) = 100x^{99} \quad \text{and} \quad g'(x) = 5$$

and so

$$\frac{d}{dx}(5x + 2)^{100} = 100(5x + 2)^{99} \cdot 5.$$

$$\text{Chain rule: } \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Example

$$\text{Calculate } \frac{d}{dx}(\sqrt{x^7 + 5}).$$

Here,

$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

So

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 7x^6$$

and so

$$\frac{d}{dx}(\sqrt{x^7 + 5}) = \frac{1}{2\sqrt{x^7 + 5}} \cdot 7x^6.$$

Derivative rules

In summary, the derivative rules we have now are

1. **Power rule:** $\frac{d}{dx}x^a = ax^{a-1}$

2. **Scalar rule:** $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x)$

3. **Sum rule:** $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

4. **Product rule:** $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

5. **Chain rule:** $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$

Examples

Use everything you know to calculate the derivatives of

1. $(3x^2 + x + 1)(5x + 1)$

2. $(3x^2 + x + 1)(5x + 1)^2$

3. $(5x + 1)^{10}$

4. $(3x^2 + x + 1)(5x + 1)^{10}$

5. $\frac{\sqrt{x^2 - x}}{x + x^{-1}}$

6. $\frac{1}{\sqrt[3]{x^2 + 7x^{1/2}}}$

Use the derivative rules (not limits) to prove the identities

a. **Reciprocal identity:** $\frac{d}{dx} \frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}$

b. **Quotient identity:** $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

c. **Many products identity:**

$$\frac{d}{dx}(f(x) * g(x) * h(x) * k(x))$$

$$= (f(x)g(x)h(x)) * k'(x) + (f(x)g(x)k(x)) * h'(x) \\ + (f(x)h(x)k(x)) * g'(x) + (g(x)h(x)k(x)) * f'(x)$$