Warmup: **Use the limit definition** of the derivative to calculate the following derivatives.

1. $\frac{d}{dx}(5x+2)$
2. $\frac{d}{dx}(3x-1)$
3. $\frac{d}{dx}\left[(5x+2)+(3x-1)\right]$
4. $\frac{d}{dx}\left[(5x+2)(3x-1)\right]$
5. $\frac{d}{dx}15x^2$
6. $\frac{d}{dx}(15x^2+x-2)$

Remember the power rule says $\frac{d}{dx}x^a = ax^{a-1}$.
Based on your calculations above, which of the following statements seem to be true and which seem to be false?

(a) If you multiply a function $f(x)$ by a number c and then take a derivative, you get the same thing as taking the derivative $f'(x)$ and then multiplying by c. (try comparing 5 to the power rule)

(b) If you add two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then adding those together. (try comparing 1-3, and then 6 to the power rule)

(c) If you multiply two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then multiplying those together. (try comparing 1, 2, and 4)

Warmup: **Use the limit definition** of the derivative to calculate the following derivatives.

1. $\frac{d}{dx}(5x+2) \;\; = 5$
2. $\frac{d}{dx}(3x-1) \;\; = 3$
3. $\frac{d}{dx}\left[(5x+2)+(3x-1)\right] \;\; = 8$
4. $\frac{d}{dx}\left[(5x+2)(3x-1)\right] = 30x+1$
5. $\frac{d}{dx}15x^2 \;\; = 30x$
6. $\frac{d}{dx}(15x^2+x-2) \;\; = 30x+1$

Remember the power rule says $\frac{d}{dx}x^a = ax^{a-1}$.
Based on your calculations above, which of the following statements seem to be true and which seem to be false?

(a) If you multiply a function $f(x)$ by a number c and then take a derivative, you get the same thing as taking the derivative $f'(x)$ and then multiplying by c. (try comparing 5 to the power rule) true?

(b) If you add two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then adding those together. (try comparing 1-3, and then 6 to the power rule) true?

(c) If you multiply two functions $f(x)$ and $g(x)$ and take a derivative, you get the same answer as taking the derivatives $f'(x)$ and $g'(x)$ and then multiplying those together. (try comparing 1, 2, and 4) false!

Multiplying by constants: what's going on?

Take another look at $f(x) = 15x^2$. Before, we just expanded and canceled, and were surprised to find something nice happened:

$$\frac{d}{dx}\left[15x^2\right] = \lim_{h \to 0} \frac{15(x+h)^2 - 15x^2}{h}$$

$$= \lim_{h \to 0} \frac{15x^2 + 30xh + 15h^2 - 15x^2}{h}$$

$$= \lim_{h \to 0} 30x + \underbrace{15h}_{\to 0} = \boxed{30x}$$

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Let's try again, only pay closer attention to that 15:

$$\frac{d}{dx}15x^2 = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h\to 0} 15 * \left(\frac{(x+h)^2 - x^2}{h}\right)$$

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$$= 15 * \underbrace{\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}}_{= \frac{d}{dx}x^2}$$

one of our limit rules! →

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But now suppose you have any differentiable function $f(x)$ and a number c. [Think: $f(x) = x^2$ and $c = 15$].

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But now suppose you have any differentiable function $f(x)$ and a number c. [Think: $f(x) = x^2$ and $c = 15$]. Then *in general*

$$\frac{d}{dx}\left(c * f(x)\right) = \lim_{h\to 0} \frac{c * f(x+h) - c * f(x)}{h}$$

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limit rule

$$\overset{*}{=} c * \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} = c * \frac{d}{dx} f(x)$$

Multiplying by constants

Theorem (Scalars)

If $y = f(x)$ is a differentiable function and c is a constant, then

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x).$$

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Example

Since $\frac{d}{dx}x^2 = 2x$, we have $\frac{d}{dx}15x^2 = 15 \cdot (2x) = 30x$.

Taking sums: what's going on?

Take another look at $f(x) = (5x+2) + (3x-1)$. Before, we just simplified first, and were surprised:

$$\frac{d}{dx}\left[(5x+2)+(3x-1)\right] = \frac{d}{dx}\left[8x+1\right] = \lim_{h\to 0}\frac{8(x+h)+1-(8x+1)}{h} = \lim_{h\to 0}\frac{8h}{h} = \boxed{8}$$

simplify first

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Let's try again, only pay closer attention to either part of the sum:

$$\frac{d}{dx}\left((5x+2)+(3x-1)\right) = \lim_{h\to 0} \frac{(5(x+h)+2)+(3(x+h)-1)-[5x+2+3x-1]}{h}$$

(go together! go together!)

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$$= \lim_{h\to 0} \frac{\left[(5(x+h)+2)-(5x+2)\right] + \left[(3(x+h)-1)-(3x-1)\right]}{h}$$

because $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

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limit rule!

$$\overset{*}{=} \lim_{h\to 0}\frac{5(x+h)+2-(5x+2)}{h}+\lim_{h\to 0}\frac{3(x+h)-1-(3x-1)}{h}$$

$$= \frac{d}{dx}(5x+2) + \frac{d}{dx}(3x-1)$$

Now, suppose you have any differentiable functions $f(x)$ and $g(x)$ [Think: $f(x) = 5x + 2$ and $g(x) = 3x - 1$].

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$$\frac{d}{dx}\left[f(x) + g(x)\right] = \lim_{h \to 0} \frac{\left(f(x+h) + g(x+h)\right) - \left(f(x) + g(x)\right)}{h}$$

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Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Example

Use the three rules we have so far

$$\frac{d}{dx}x^a = ax^{a-1}, \quad \frac{d}{dx}(c \cdot f(x)) = c \cdot \left(\frac{d}{dx}f(x)\right),$$

$$\text{and} \quad \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

to calculate the derivatives:

1. $\frac{d}{dx}(x^3 - 7x^2 + 6x^{-15})$

2. $\frac{d}{dx}\left(\sqrt{x} + 100\sqrt[17]{x^3} - \frac{3}{x^{19}}\right)$

[hint: rewrite everything from 2 as powers before you do anything]

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to calculate the derivatives:

1. $\frac{d}{dx}(x^3 - 7x^2 + 6x^{-15})$

   $= \frac{d}{dx}x^3 - 7 \cdot \frac{d}{dx}x^2 + 6 \cdot \frac{d}{dx}x^{-15} = \boxed{3x^2 - 7 \cdot (2x) + 6(-15)x^{-16}}$

2. $\frac{d}{dx}\left(\sqrt{x} + 100\sqrt[17]{x^3} - \frac{3}{x^{19}}\right) = \frac{d}{dx}\left(x^{1/2} + 100x^{3/17} - 3x^{-19}\right)$
   $= \frac{d}{dx}x^{1/2} + 100 \cdot \frac{d}{dx}x^{3/17} - 3 \cdot \frac{d}{dx}x^{-19}$
   $= \boxed{\frac{1}{2}x^{-1/2} + 100 \cdot \left(\frac{3}{17}\right)x^{-14/17} - 3 \cdot (-19)x^{-20}}$

[hint: rewrite everything from 2 as powers before you do anything]

Products: What's going on?

Take another look at $f(x) = (5x+2)\cdot(3x-1)$. Before, we just simplified first, and were. . . not surprised:

$$\frac{d}{dx}\left[(5x+2)(3x-1)\right] = \frac{d}{dx}\left(15x^2 + x - 2\right)$$

$$= \lim_{h\to 0} \frac{15(x+h)^2 + (x+h) - 2 - (15x^2 + x - 2)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(15x^2 + 30xh + 15h^2 + x + h - 2 - 15x^2 - x + 2\right)$$

$$= \lim_{h\to 0} 30x + 15h + 1 = \boxed{30x+1}$$

We *didn't* get that the derivative of the products is the product of the derivatives! So what *is* going on here?

To understand how to deal with products, we're going to have to unpack the formula

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

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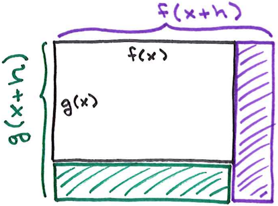

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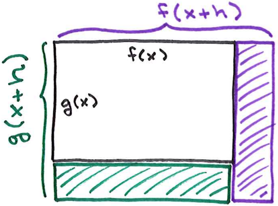

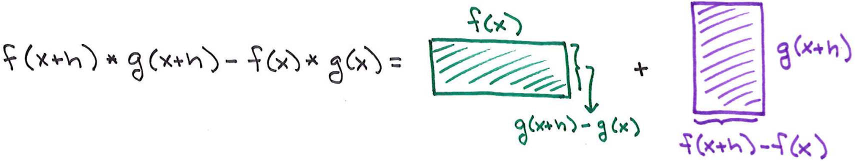

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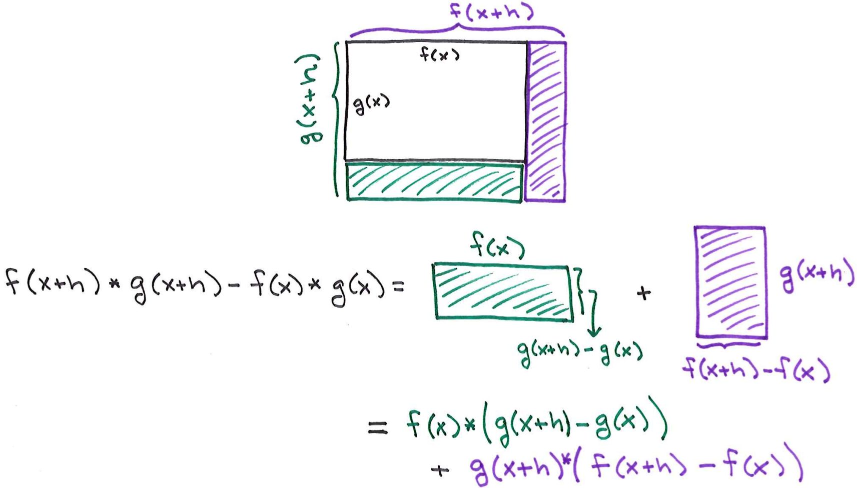
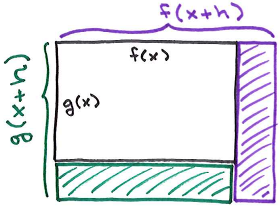
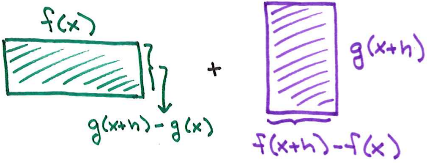
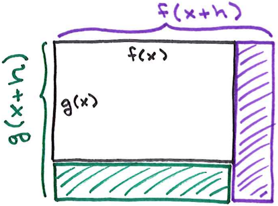




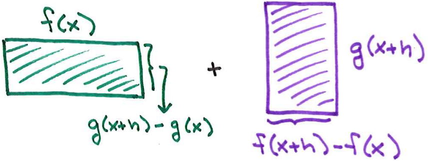
$$f(x+h) * g(x+h) - f(x) * g(x) =$$



$$= f(x) * \big(g(x+h) - g(x)\big) + g(x+h) * \big(f(x+h) - f(x)\big)$$



$$f(x+h)*g(x+h) - f(x)*g(x) =$$



$$= f(x)*\big(g(x+h)-g(x)\big) + g(x+h)*\big(f(x+h)-f(x)\big)$$

So

$$\frac{d}{dx} f(x)*g(x) = \lim_{h\to 0} \frac{f(x+h)*g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\Big(f(x)*[g(x+h)-g(x)] + g(x+h)[f(x+h)-f(x)] \Big)$$

So

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$$= \lim_{h \to 0} \frac{1}{h} \left(f(x) * [g(x+h) - g(x)] + g(x+h)[f(x+h) - f(x)] \right)$$

$$= \lim_{h \to 0} \left[f(x) * \left(\frac{g(x+h) - g(x)}{h} \right) + g(x+h) * \left(\frac{f(x+h) - f(x)}{h} \right) \right]$$

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$$\frac{d}{dx} f(x) * g(x) = \lim_{h \to 0} \frac{f(x+h) * g(x+h) - f(x) g(x)}{h}$$

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$$= f(x) * \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \rightarrow g'(x)$$

$$+ \left(\lim_{h \to 0} g(x+h) \right) * \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right)$$

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Theorem (Products)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

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Example: Calculate $\frac{d}{dx}\big((5x+2)(3x-1)\big)$:

$$\frac{d}{dx}\big(\underset{f\,\uparrow}{(5x+2)}\underset{g\,\uparrow}{(3x-1)}\big) = \underset{f}{(5x+2)}\cdot\underset{g'}{3}+\underset{g}{(3x-1)}\cdot\underset{f'}{5} = \boxed{\text{30x+1}} \;☺$$

Last rule: Compositions.

Example: Calculate $\frac{d}{dx}(5x+2)^{100}$.

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So since $f'(x) = 100x^{99}$ and $g'(x) = 5$, if everything were easy in the world, we might hope that

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Theorem (Chain rule)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}((f\circ g)(x))=\frac{d}{dx}\big(f(g(x))\big)=f'(g(x))\cdot g'(x).$$

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$= f'(g(x))$

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$\frac{g(x+h) - g(x)}{h} \to g'(x)$

In Leibniz notation:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\,(5x+2)^{100}$.

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Example

Calculate $\frac{d}{dx}\,(5x+2)^{100}$.

Here,

$$f(x) = x^{100} \quad \text{and} \quad g(x) = 5x + 2.$$

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and so

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$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

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$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

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Derivative rules

In summary, the derivative rules we have now are

1. **Power rule:** $\frac{d}{dx}x^a = ax^{a-1}$
2. **Scalar rule:** $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x)$
3. **Sum rule:** $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
4. **Product rule:** $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
5. **Chain rule:** $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$

Examples

Use everything you know to calculate the derivatives of

1. $(3x^2 + x + 1)(5x + 1)$
2. $(3x^2 + x + 1)(5x + 1)^2$
3. $(5x + 1)^{10}$
4. $(3x^2 + x + 1)(5x + 1)^{10}$
5. $\dfrac{\sqrt{x^2 - x}}{x + x^{-1}}$
6. $\dfrac{1}{\sqrt[3]{x^2 + 7x^{1/2}}}$

Use the derivative rules (not limits) to prove the identities

a. **Reciprocal identity**: $\dfrac{d}{dx}\dfrac{1}{f(x)} = -\dfrac{f'(x)}{f^2(x)}$

b. **Quotient identity**: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

c. **Many products identity**:

$$\frac{d}{dx}(f(x) * g(x) * h(x) * k(x)) = \Big(f(x)g(x)h(x)\Big) * k'(x) + \Big(f(x)g(x)k(x)\Big) * h'(x) + \Big(f(x)h(x)k(x)\Big) * g'(x) + \Big(g(x)h(x)k(x)\Big) * f'(x)$$