

**Warmup:** Use the limit definition of the derivative to calculate the following derivatives.

1.  $\frac{d}{dx}(5x + 2)$

2.  $\frac{d}{dx}(3x - 1)$

3.  $\frac{d}{dx}[(5x + 2) + (3x - 1)]$

4.  $\frac{d}{dx}[(5x + 2)(3x - 1)]$

5.  $\frac{d}{dx}15x^2$

6.  $\frac{d}{dx}(15x^2 + x - 2)$

Remember the power rule says  $\frac{d}{dx}x^a = ax^{a-1}$ .

Based on your calculations above, which of the following statements seem to be true and which seem to be false?

- (a) If you multiply a function  $f(x)$  by a number  $c$  and then take a derivative, you get the same thing as taking the derivative  $f'(x)$  and then multiplying by  $c$ . (try comparing 5 to the power rule)
- (b) If you add two functions  $f(x)$  and  $g(x)$  and take a derivative, you get the same answer as taking the derivatives  $f'(x)$  and  $g'(x)$  and then adding those together. (try comparing 1-3, and then 6 to the power rule)
- (c) If you multiply two functions  $f(x)$  and  $g(x)$  and take a derivative, you get the same answer as taking the derivatives  $f'(x)$  and  $g'(x)$  and then multiplying those together. (try comparing 1, 2, and 4)

**Warmup:** Use the limit definition of the derivative to calculate the following derivatives.

$$1. \frac{d}{dx}(5x + 2) = 5$$

$$4. \frac{d}{dx}[(5x+2)(3x-1)] = 30x + 1$$

$$2. \frac{d}{dx}(3x - 1) = 3$$

$$5. \frac{d}{dx}15x^2 = 30x$$

$$3. \frac{d}{dx}[(5x + 2) + (3x - 1)] = 8$$

$$6. \frac{d}{dx}(15x^2 + x - 2) = 30x + 1$$

Remember the power rule says  $\frac{d}{dx}x^a = ax^{a-1}$ .

Based on your calculations above, which of the following statements seem to be true and which seem to be false?

- (a) If you multiply a function  $f(x)$  by a number  $c$  and then take a derivative, you get the same thing as taking the derivative  $f'(x)$  and then multiplying by  $c$ . (try comparing 5 to the power rule) **true?**
- (b) If you add two functions  $f(x)$  and  $g(x)$  and take a derivative, you get the same answer as taking the derivatives  $f'(x)$  and  $g'(x)$  and then adding those together. (try comparing 1-3, and then 6 to the power rule) **true?**
- (c) If you multiply two functions  $f(x)$  and  $g(x)$  and take a derivative, you get the same answer as taking the derivatives  $f'(x)$  and  $g'(x)$  and then multiplying those together. (try comparing 1, 2, and 4) **false!**

## Multiplying by constants: what's going on?

Take another look at  $f(x) = 15x^2$ . Before, we just expanded and canceled, and were surprised to find something nice happened:

$$\begin{aligned}\frac{d}{dx} [15x^2] &= \lim_{h \rightarrow 0} \frac{15(x+h)^2 - 15x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{15x^2 + 30xh + 15h^2 - 15x^2}{h} \\ &= \lim_{h \rightarrow 0} 30x + \frac{15h}{h} = \boxed{30x}\end{aligned}$$

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Let's try again, only pay closer attention to that 15:

$$\frac{d}{dx} 15x^2 = \lim_{h \rightarrow 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h \rightarrow 0} 15 * \frac{(x+h)^2 - x^2}{h}$$

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Let's try again, only pay closer attention to that 15:

$$\begin{aligned}\frac{d}{dx} 15x^2 &= \lim_{h \rightarrow 0} \frac{15(x+h)^2 - 15x^2}{h} = \lim_{h \rightarrow 0} 15 * \frac{((x+h)^2 - x^2)}{h} \\ &= 15 * \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \frac{d}{dx} x^2\end{aligned}$$

one of our limit rules!

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But now suppose you have any differentiable function  $f(x)$  and a number  $c$ . [Think:  $f(x) = x^2$  and  $c = 15$ ].

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But now suppose you have any differentiable function  $f(x)$  and a number  $c$ . [Think:  $f(x) = x^2$  and  $c = 15$ ]. Then *in general*

$$\frac{d}{dx} (c * f(x)) = \lim_{h \rightarrow 0} \frac{c * f(x+h) - c * f(x)}{h}$$

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Let's try again, only pay closer attention to that 15:

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# Multiplying by constants

## Theorem (Scalars)

*If  $y = f(x)$  is a differentiable function and  $c$  is a constant, then*

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x).$$

# Multiplying by constants

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## Example

Since  $\frac{d}{dx}x^2 = 2x$ , we have  $\frac{d}{dx}15x^2 = 15 \cdot (2x) = 30x$ .

## Taking sums: what's going on?

Take another look at  $f(x) = (5x + 2) + (3x - 1)$ . Before, we just simplified first, and were surprised:

$$\begin{aligned} \frac{d}{dx} [(5x+2) + (3x-1)] &= \frac{d}{dx} [8x + 1] = \lim_{h \rightarrow 0} \frac{8(x+h) + 1 - (8x+1)}{h} \\ &\quad \xrightarrow{\text{simplify first}} \\ &= \lim_{h \rightarrow 0} \frac{8h}{h} = \boxed{8} \end{aligned}$$

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*simplify first*

Let's try again, only pay closer attention to either part of the sum:

$$\frac{d}{dx} \left( \underline{(5x+2)} + (3x-1) \right) = \lim_{h \rightarrow 0} \frac{\overbrace{(5(x+h)+2)}^{\text{go together!}} + \overbrace{(3(x+h)-1)}^{\text{go together!}} - [5x+2+3x-1]}{h}$$

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Let's try again, only pay closer attention to either part of the sum:

$$\begin{aligned} \frac{d}{dx} ((5x+2) + (3x-1)) &= \lim_{h \rightarrow 0} \frac{(5(x+h)+2) + (3(x+h)-1) - [5x+2 + 3x-1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(5(x+h)+2) - (5x+2)] + [(3(x+h)-1) - (3x-1)]}{h} \end{aligned}$$

*go together!*      *go together!*

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because  $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

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*simplify first*

Let's try again, only pay closer attention to either part of the sum:

$$\begin{aligned} \frac{d}{dx} ((5x+2) + (3x-1)) &= \lim_{h \rightarrow 0} \frac{(5(x+h)+2) + (3(x+h)-1) - [5x+2+3x-1]}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{(5(x+h)+2) - (5x+2)}{h} + \frac{(3(x+h)-1) - (3x-1)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{[5(x+h)+2] - (5x+2)}{h} + \lim_{h \rightarrow 0} \frac{[3(x+h)-1] - (3x-1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5(x+h)+2 - (5x+2)}{h} + \lim_{h \rightarrow 0} \frac{3(x+h)-1 - (3x-1)}{h} \\ &= \frac{d}{dx} (5x+2) + \frac{d}{dx} (3x-1) \end{aligned}$$

*because  $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$*

*limit rule!*



Now, suppose you have any differentiable functions  $f(x)$  and  $g(x)$   
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$$\underline{\underline{\frac{d}{dx} [f(x) + g(x)]}} = \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

Diagram illustrating the limit process for the derivative of a sum. The expression is  $\lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$ . Two curved arrows point from the terms in the numerator to the terms in the denominator:

- A purple arrow points from  $f(x+h)$  to  $f(x)$  with the text "go together!".
- A green arrow points from  $g(x+h)$  to  $g(x)$  with the text "go together!".

Now, suppose you have any differentiable functions  $f(x)$  and  $g(x)$   
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*go together!*      *go together!*

*(because  $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ )*

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*Annotations:*  
- Purple arrows labeled "go together!" point from the sum in the numerator to each term in the denominator of the second line.  
- Green arrows labeled "go together!" point from the sum in the numerator to each term in the denominator of the second line.  
- A red arrow labeled "limit rule!" points to the asterisk in the third line.

Now, suppose you have any differentiable functions  $f(x)$  and  $g(x)$   
[Think:  $f(x) = 5x + 2$  and  $g(x) = 3x - 1$ ]. Then *in general*

$$\begin{aligned} \underline{\underline{\frac{d}{dx} [f(x) + g(x)]}} &= \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \quad \left( \text{because } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \underline{\underline{\frac{d}{dx} f(x) + \frac{d}{dx} g(x)}} \end{aligned}$$

*Annotations:*  
- Purple arrows labeled "go together!" point from the sum in the numerator to the separate terms in the denominator.  
- Green arrows labeled "go together!" point from the sum in the numerator to the separate terms in the denominator.  
- Red arrow labeled "limit rule!" points to the limit operation in the second step.  
- Red asterisk (\*) is next to the limit operation in the second step.

Now, suppose you have any differentiable functions  $f(x)$  and  $g(x)$   
 [Think:  $f(x) = 5x + 2$  and  $g(x) = 3x - 1$ ]. Then *in general*

$$\begin{aligned}
 \frac{d}{dx} [f(x) + g(x)] &= \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \quad (\text{because } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}) \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
 \end{aligned}$$

*Handwritten notes:*  
 - Purple arrows labeled "go together!" point from the sum in the numerator to the separate terms in the denominator.  
 - A red arrow labeled "limit rule!" points to the limit operation in the second step.  
 - The final result is underlined in orange.

## Theorem (Sums)

If  $f$  and  $g$  are differentiable functions, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

## Example

Use the three rules we have so far

$$\frac{d}{dx}x^a = ax^{a-1}, \quad \frac{d}{dx}(c \cdot f(x)) = c \cdot \left(\frac{d}{dx}f(x)\right),$$

$$\text{and} \quad \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

to calculate the derivatives:

1.  $\frac{d}{dx}(x^3 - 7x^2 + 6x^{-15})$
  
  
  
  
  
  
  
  
  
  
2.  $\frac{d}{dx}\left(\sqrt{x} + 100\sqrt[17]{x^3} - \frac{3}{x^{19}}\right)$

[hint: rewrite everything from 2 as powers before you do anything]

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$$\frac{d}{dx}x^a = ax^{a-1}, \quad \frac{d}{dx}(c \cdot f(x)) = c \cdot \left(\frac{d}{dx}f(x)\right),$$

$$\text{and} \quad \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

to calculate the derivatives:

1.  $\frac{d}{dx}(x^3 - 7x^2 + 6x^{-15})$

$$= \frac{d}{dx}x^3 - 7 \cdot \frac{d}{dx}x^2 + 6 \cdot \frac{d}{dx}x^{-15} = \boxed{3x^2 - 7 \cdot (2x) + 6(-15)x^{-16}}$$

2.  $\frac{d}{dx} \left( \sqrt{x} + 100 \sqrt[17]{x^3} - \frac{3}{x^{19}} \right) = \frac{d}{dx} \left( x^{1/2} + 100x^{3/17} - 3x^{-19} \right)$

$$= \frac{d}{dx}x^{1/2} + 100 \cdot \frac{d}{dx}x^{3/17} - 3 \cdot \frac{d}{dx}x^{-19}$$

$$= \boxed{\frac{1}{2}x^{-1/2} + 100 \cdot \left(\frac{3}{17}\right)x^{-14/17} - 3 \cdot (-19)x^{-20}}$$

[hint: rewrite everything from 2 as powers before you do anything]



## Products: What's going on?

Take another look at  $f(x) = (5x + 2) \cdot (3x - 1)$ . Before, we just simplified first, and were... not surprised:

$$\begin{aligned}\frac{d}{dx}[(5x+2)(3x-1)] &= \frac{d}{dx}(15x^2 + x - 2) \\ &= \lim_{h \rightarrow 0} \frac{15(x+h)^2 + (x+h) - 2 - (15x^2 + x - 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (15x^2 + \underline{30xh} + \underline{15h^2} + x + \underline{h} - 2 - 15x^2 - x + 2) \\ &= \lim_{h \rightarrow 0} 30x + \underline{15h} + 1 = \boxed{30x + 1}\end{aligned}$$

We *didn't* get that the derivative of the products is the product of the derivatives! So what *is* going on here?

To understand how to deal with products, we're going to have to unpack the formula

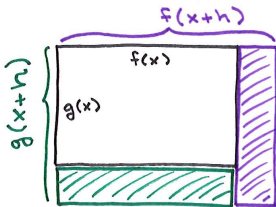
$$\frac{d}{dx} (f(x) \cdot g(x)) = \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

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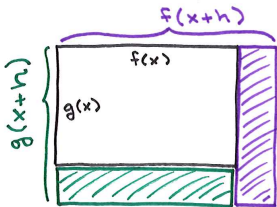
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$$\frac{d}{dx} (f(x) \cdot g(x)) = \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

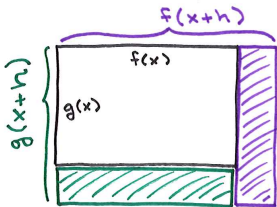


$$f(x+h) \cdot g(x+h) - f(x) \cdot g(x) = \underbrace{f(x)}_{\text{width}} \underbrace{g(x+h) - g(x)}_{\text{height}} + \underbrace{f(x+h) - f(x)}_{\text{width}} \underbrace{g(x+h)}_{\text{height}}$$

The diagram shows two hatched rectangles. The first is green, with a width labeled  $f(x)$  and a height labeled  $g(x+h) - g(x)$ . The second is purple, with a width labeled  $f(x+h) - f(x)$  and a height labeled  $g(x+h)$ . A plus sign is between them.

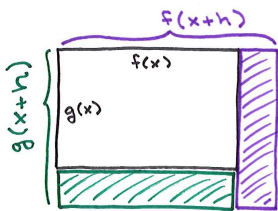
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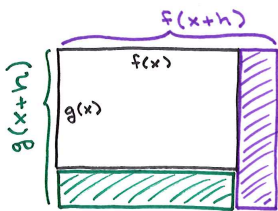


$$f(x+h) \cdot g(x+h) - f(x) \cdot g(x) = \underbrace{\hspace{10em}}_{g(x+h) - g(x)} + \underbrace{\hspace{10em}}_{f(x+h) - f(x)}$$

$$= f(x) \cdot (g(x+h) - g(x)) + g(x+h) \cdot (f(x+h) - f(x))$$



$$\begin{aligned}
 f(x+h) * g(x+h) - f(x) * g(x) &= \underbrace{\text{[Green shaded rectangle]}}_{g(x+h) - g(x)} + \underbrace{\text{[Purple shaded rectangle]}}_{f(x+h) - f(x)} \\
 &= f(x) * (g(x+h) - g(x)) \\
 &\quad + g(x+h) * (f(x+h) - f(x))
 \end{aligned}$$



$$\begin{aligned}
 f(x+h) * g(x+h) - f(x) * g(x) &= \underbrace{\hspace{10em}}_{f(x)} \underbrace{\hspace{10em}}_{g(x+h) - g(x)} + \underbrace{\hspace{10em}}_{f(x+h) - f(x)} \underbrace{\hspace{10em}}_{g(x+h)} \\
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So

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 \frac{d}{dx} f(x) * g(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) * g(x+h) - f(x) * g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( f(x) * [g(x+h) - g(x)] + g(x+h) [f(x+h) - f(x)] \right)
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$$= f(x) * \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \rightarrow g'(x)$$

$$+ \left( \lim_{h \rightarrow 0} g(x+h) \right) * \left( \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right)$$

$\downarrow$   $g(x)$                        $\downarrow$   $f'(x)$

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## Theorem (Products)

*If  $f(x)$  and  $g(x)$  are differentiable functions, then*

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

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**Example:** Calculate  $\frac{d}{dx}((5x + 2)(3x - 1))$ :

$$\frac{d}{dx}((5x + 2)(3x - 1)) = (5x + 2) \cdot 3 + (3x - 1) \cdot 5 = \boxed{30x+1} \text{ ☺}$$

$f \uparrow \quad g \uparrow \quad f \cdot g' + g \cdot f'$

## Last rule: Compositions.

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$\downarrow$   
how  $f$  changes versus  $g(x)$  (instead of versus  $x$ )

$\downarrow$   
 $g'(x)$

$$= f'(g(x))$$

In Leibniz notation:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Chain rule:  $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$ .

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Calculate  $\frac{d}{dx}(5x + 2)^{100}$ .

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and so

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### Example

Calculate  $\frac{d}{dx}(\sqrt{x^7 + 5})$ .

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Calculate  $\frac{d}{dx}(\sqrt{x^7 + 5})$ .

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$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

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### Example

Calculate  $\frac{d}{dx}(\sqrt{x^7 + 5}).$

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So

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 7x^6$$



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# Derivative rules

In summary, the derivative rules we have now are

1. **Power rule:**  $\frac{d}{dx}x^a = ax^{a-1}$

2. **Scalar rule:**  $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x)$

3. **Sum rule:**  $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

4. **Product rule:**  $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

5. **Chain rule:**  $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$

## Examples

Use everything you know to calculate the derivatives of

1.  $(3x^2 + x + 1)(5x + 1)$

2.  $(3x^2 + x + 1)(5x + 1)^2$

3.  $(5x + 1)^{10}$

4.  $(3x^2 + x + 1)(5x + 1)^{10}$

5.  $\frac{\sqrt{x^2 - x}}{x + x^{-1}}$

6.  $\frac{1}{\sqrt[3]{x^2 + 7x^{1/2}}}$

Use the derivative rules (not limits) to prove the identities

a. **Reciprocal identity:**  $\frac{d}{dx} \frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}$

b. **Quotient identity:**  $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

c. **Many products identity:**

$$\begin{aligned} & \frac{d}{dx} (f(x) * g(x) * h(x) * k(x)) \\ &= \left( f(x)g(x)h(x) \right) * k'(x) + \left( f(x)g(x)k(x) \right) * h'(x) \\ &+ \left( f(x)h(x)k(x) \right) * g'(x) + \left( g(x)h(x)k(x) \right) * f'(x) \end{aligned}$$