Today: Derivatives of a function

Warmup:

1. Let $f(x) = \sqrt{x}$. Compute the following limits.
   (a) $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$
   (b) $\lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$
   (c) $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, where a is non-negative real number.
2. Let $f(x) = x^2 + 3x$. Compute the following limits.
   (a) $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$
   (b) $\lim_{h \to 0} \frac{f(-5+h) - f(-5)}{h}$
   (c) $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, where a is non-negative real number.

Answers: 1(a) $1/2$, (b) $1/2\sqrt{5}$, (c) $1/2\sqrt{a}$; 2(a) 9, (b) -7, (c) $2a + 3$.

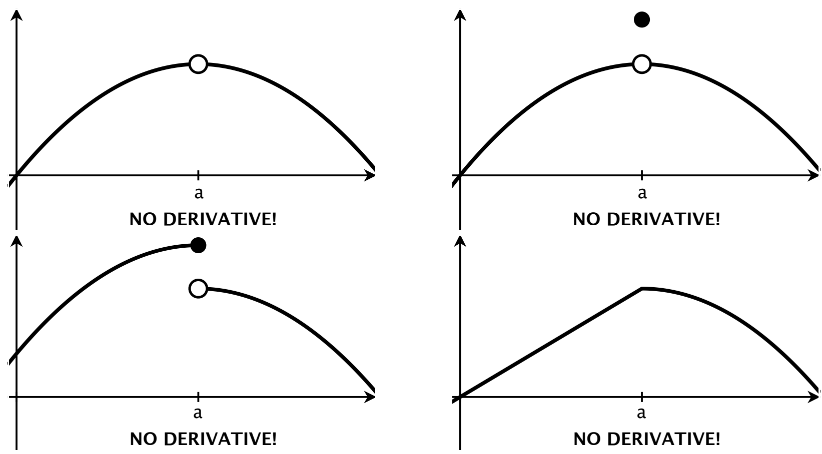
Definition

The **derivative** of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will say that a function f is **differentiable** over an interval (a, b) if the derivative function $f'(x)$ exists at every point in (a, b).

Remember from last time:



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(see notes from last time for when derivatives *don't* exist)

Q. How is this the same or different from what we were doing last time with tangent lines?

A. Last time, we were calculating derivatives at individual points, and getting **numbers** for answers. Today, we'll calculate the **derivative function**, and get out answers with variables in them (do all the points at once).

Example: let $f(x) = x^2$

Derivatives at a point: If I first ask "what is $f'(2)$?", I could calculate

$$f'(2) = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} 4 + h = \boxed{4}.$$

But then, if I ask "what is $f'(3)$?" we have to do it all over again.

Today's goal: Write down a **function** $f'(x)$ which has all the derivatives-at-a-point collected together.

If a is a number, (like 2 or 3) then

$$f'(a) = \underbrace{\lim_{h\to 0}}_{\text{gets rid of the } h\text{'s}} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\text{a function of } a\text{'s and } h\text{'s}} = \text{number}$$

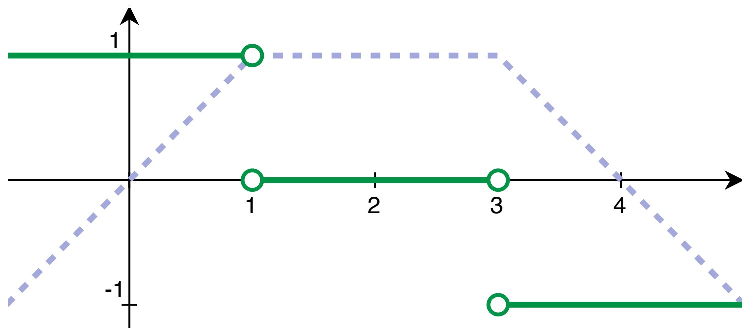
But x is a variable, so

$$f'(x) = \underbrace{\lim_{h\to 0}}_{\text{gets rid of the } h\text{'s}} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{a function of } x\text{'s and } h\text{'s}} = \text{function of } x\text{'s}$$

Starting simple

Suppose we consider the piecewise linear function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x+4 & 3 \leq x \end{cases}$$



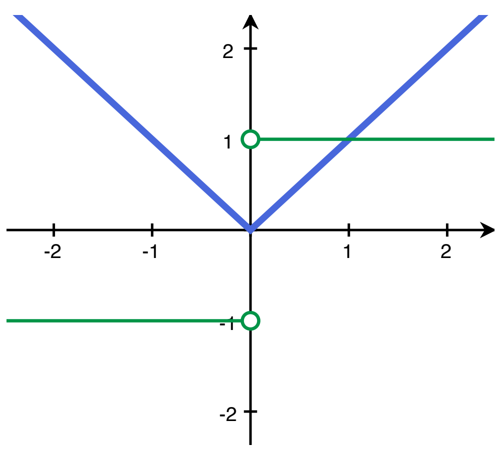
The derivative is:

$$f'(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

Another example

What is the derivative of $f(x) = |x|$?

Write down the piecewise function and sketch it on the graph.

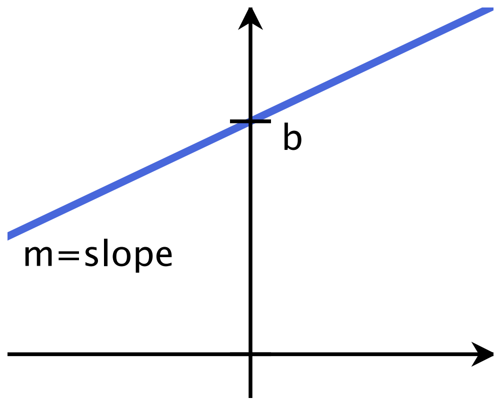


$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Lines

In general, if m and b are constants, and

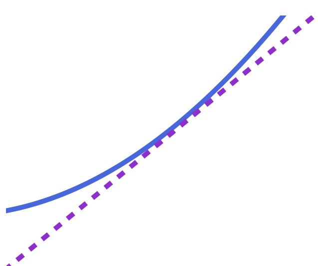
$$f(x) = mx + b$$



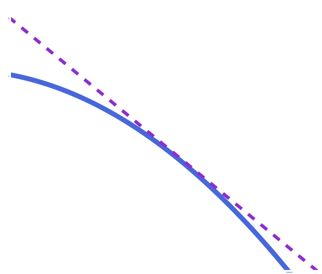
$$f'(x) = m$$

the slope of the tangent line = slope of the line

Rough shape of the derivative



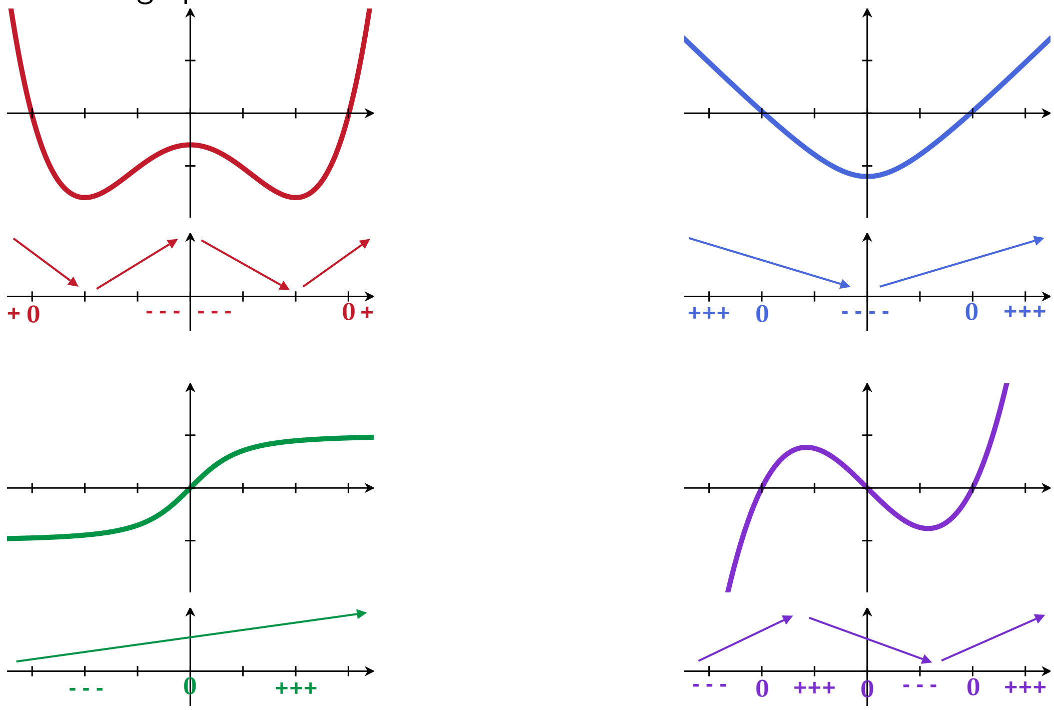
if $f(x)$ is increasing
$f'(x)$ is positive!



if $f(x)$ is decreasing
$f'(x)$ is negative!

Match em up!

Here are graphs of two functions and their derivatives. Which are which?

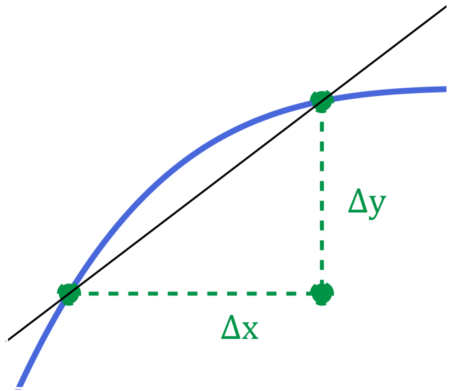


A little more notation

Back to what $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ means:

Rename $\boxed{h = \Delta x}$ and $\boxed{f(x+h) - f(x) = \Delta y}$

(Δ means "change")



So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

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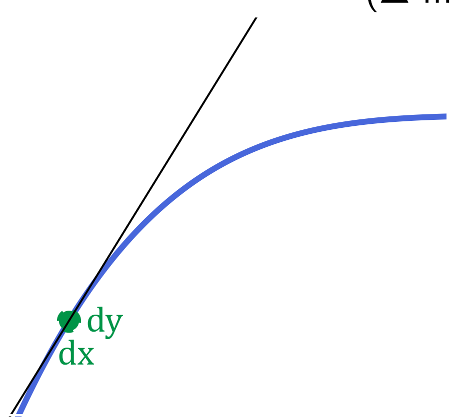

As $h \to 0$,
Δx and Δy get infinitely small.

$$\Delta x \rightsquigarrow dx \qquad \Delta y \rightsquigarrow dy$$

$$\frac{\Delta y}{\Delta x} \rightsquigarrow \frac{dy}{dx}$$

"infinitesimals"

So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

Leibniz notation

One way to write the derivative of $f(x)$ versus x is $f'(x)$.
Another way to write it is

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}f(x).$$

Derivatives at a point: $f'(a)$ means the derivative of $f(x)$ **evaluated at a**. Another way to write it is

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}$$

Example: We can write the derivative of x^2 as $\frac{d}{dx}x^2$

and the derivative of x^2 at $x = 5$ as $\left.\frac{d}{dx}x^2\right|_{x=5}$

Other notation:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = D(f)(x) = D_x f(x).$$

Go to work: building our first derivative rule.

Example 1: What *is* the derivative of x^2?

$$\begin{aligned}\frac{d}{dx}x^2 &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h} \\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} 2x+h = \boxed{2x} \qquad \left(\text{so } \left.\frac{d}{dx}x^2\right|_{x=5} = 2*5\right)\end{aligned}$$

By taking limits, fill in the rest of the table:

$f(x)$	1	x	x^2	x^3	$\frac{1}{x}$	$\frac{1}{x^2}$	$\sqrt{x}$	$\sqrt[3]{x}$
$f'(x)$			$2x$					

Hints: For $\frac{1}{x^2}$, find a common denominator, and then expand.
For $\sqrt[3]{x}$, try multiplying and dividing by $(\sqrt[3]{x+h})^2 + (\sqrt[3]{x+h})(\sqrt[3]{x}) + (\sqrt[3]{x})^2$.

$f(x)$	$f'(x)$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x} = x^{-1}$	$-x^{-2}$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3}$
$\sqrt{x} = x^{1/2}$	$(1/2)x^{-1/2}$
$\sqrt[3]{x} = x^{1/3}$	$(1/3)x^{-2/3}$

Power rule:

$$\frac{d}{dx}x^a = ax^{a-1}$$

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2}x^{3/2}} \quad \textit{1}^{\textit{st}} \textit{ derivative} = f'(x) = \frac{d}{dx}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2}x^{1/2}} \quad \textit{2}^{\textit{nd}} \textit{ derivative} = f''(x) = \frac{d^2}{dx^2}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2}x^{-1/2}} \quad \textit{3}^{\textit{rd}} \textit{ derivative} = f^{(3)}(x) = \frac{d^3}{dx^3}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \left(-\frac{1}{2}\right) x^{-3/2}}$$

$$\textit{4}^{\textit{th}} \textit{ derivative} = f^{(4)}(x) = \frac{d^4}{dx^4}x^2$$

The n^{th} derivative of $f(x)$ is denoted

$$\underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n} f(x) = \frac{d^n}{dx^n}f(x) = f^{(n)}(x).$$