Today: Tangent Lines and the Derivative at a Point

Warmup:

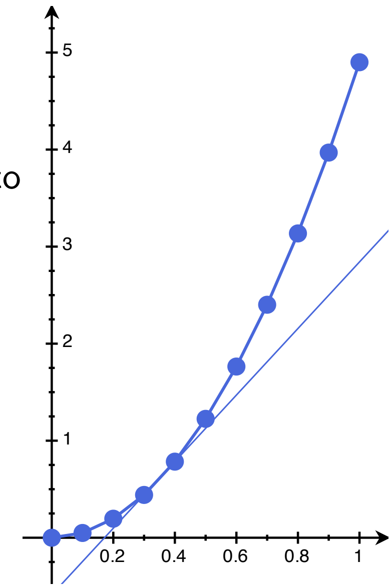
1. Let $f(x) = x^2$. Compute the following limits.
   (a) $\lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$
   (b) $\lim\limits_{h\to 0} \dfrac{f(-2+h)-f(-2)}{h}$
2. Let $g(x) = \frac{1}{x}$. Compute the following limits.
   (a) $\lim\limits_{h\to 0} \dfrac{g(3+h)-g(3)}{h}$
   (b) $\lim\limits_{h\to 0} \dfrac{g(-5+h)-g(-5)}{h}$

Answers: 1(a) 2, (b) -8; 2(a) $-\frac{1}{9}$, (b) $-\frac{1}{25}$

Recall: Average velocity

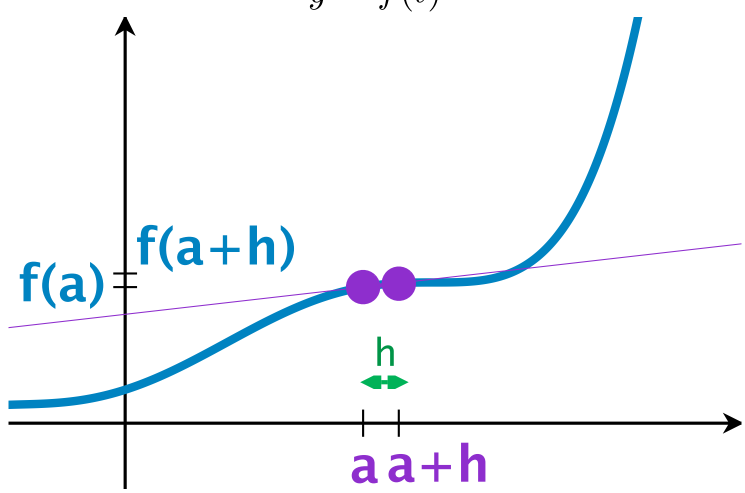
The **average velocity** from $t = t_1$ to $t = t_2$ is

$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \text{slope of secant line}$$



Plot position f versus time t:

$y = f(t)$



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.
Slope of the line connecting them:

$$\text{avg velocity} = m = \frac{f(a+h) - f(a)}{h} \qquad \text{"difference quotient"}$$

The smaller h is, the more useful m is!

The **difference quotient** (average velocity) of the curve $y = f(x)$ at $x = a$ is

$$\text{avg velocity} = \frac{f(a+h) - f(a)}{h}.$$

The **slope** (instantaneous velocity) of the curve $y = f(x)$ at the point $(a, f(a))$, called the **derivative of $f(x)$ at $x = a$**, is the number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(provided the limit exists).

The **tangent line** ℓ to the curve at $(a, f(a))$ is the line through $(a, f(a))$ with this slope:

$$\ell: \; y - f(a) = m(x - a), \quad \text{where } m = f'(a).$$

(Recall point-slope form: $y - y_0 = m(x - x_0)$.)

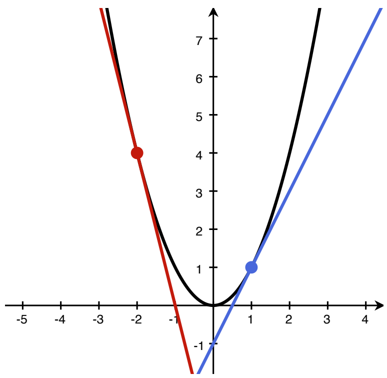
The **slope** (instantaneous velocity) of the curve $y = f(x)$ at the point $(a, f(a))$, called the **derivative of $f(x)$ at $x = a$**, is the number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(provided the limit exists).

Recall the warmup:

1. Let $f(x) = x^2$. Compute the following limits.
   (a) $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \boxed{2}$ ← slope at $x = 1$
   (b) $\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \boxed{-4}$ ← slope at $x = 2$



blue line: $y - 1 = 2(x - 1)$

red line: $y - 4 = -4(x + 2)$

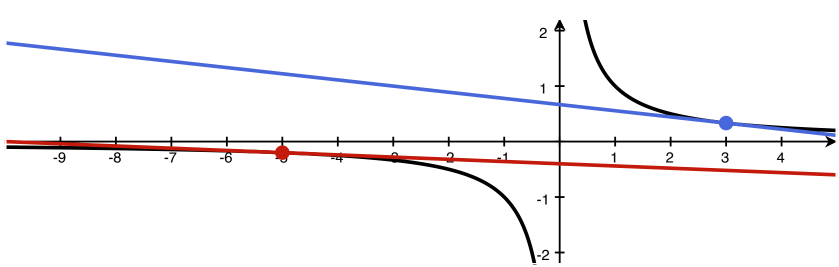
The **slope** (instantaneous velocity) of the curve $y = f(x)$ at the point $(a, f(a))$, called the **derivative of $f(x)$ at $x = a$**, is the number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(provided the limit exists).

Recall the warmup:

2. Let $g(x) = \frac{1}{x}$. Compute the following limits.
   (a) $\lim_{h \to 0} \frac{g(3+h) - g(3)}{h} = \boxed{-1/9}$ ← slope at $x = 3$
   (b) $\lim_{h \to 0} \frac{g(-5+h) - g(-5)}{h} = \boxed{-1/25}$ ← slope at $x = -5$



blue line: $y - \frac{1}{3} = -\frac{1}{9}(x - 3)$ red line: $y + \frac{1}{5} = -\frac{1}{25}(x + 5)$

You try:

For each of the following examples...

(a) Compute $f'(a)$ using the formula

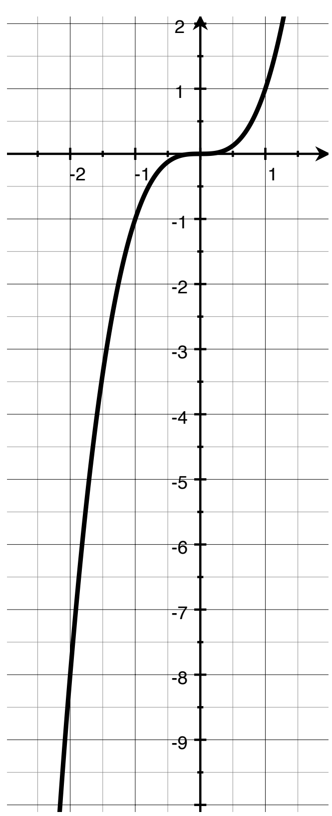
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

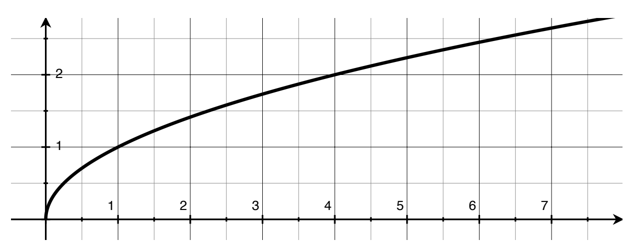
(b) Compute the equation for the tangent line to $(a, f(a))$ using point-slope form.

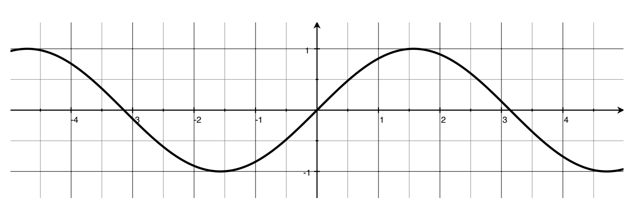
(c) Sketch $y = f(x)$ near $x = a$ and the line you computed in part (b) on the same set of axes to check that your answers make sense.

1. $f(x) = x^3$ at $a = 0$, $a = 1$, and $a = -2$.
2. $f(x) = \sqrt{x}$ at $a = 1$ and $a = 4$.
3. $f(x) = \sin(x)$ at $a = 0$, $a = \pi/4$ and $a = -\pi/2$.

[For 3, recall $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ and $\lim_{\theta \to 0} \sin(\theta)/\theta = 1$.]





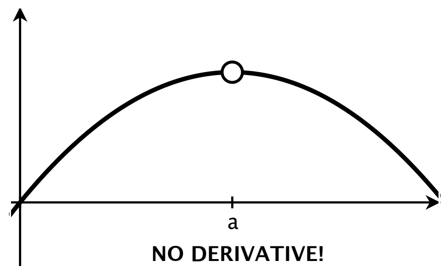


When can we take derivatives?

Not all functions have derivatives at all places. Before calculating $f'(a)$, first ask ...

(1) Is $f(x)$ defined at $x = a$?

For example, even if it looks like you could draw a tangent line, if there's a hole, $f'(a)$ **does not exist**



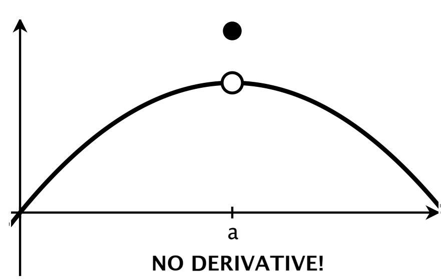
(It's tempting to say $f'(a)$ exists here in part because $f(x)$ has a *continuous extension* at a.)

When can we take derivatives?

Not all functions have derivatives at all places. Before calculating $f'(a)$, first ask ...

(2) Is $f(x)$ continuous at $x = a$?

For example, even if it looks like you could draw a tangent line, if there's a jump, $f'(a)$ **does not exist**!



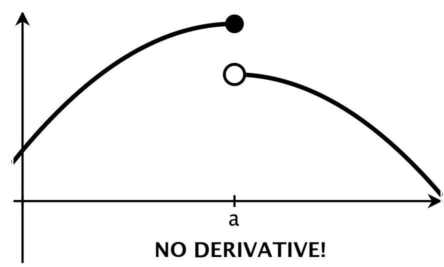
(Try drawing just one line that is tangent to that isolated point. It's tempting to say $f'(a)$ exists here in part because $f(x)$ has a *removable discontinuity* at a.)

When can we take derivatives?

Not all functions have derivatives at all places. Before calculating $f'(a)$, first ask ...

(2) Is $f(x)$ continuous at $x = a$?

Again, even if the slope looks the same from the left and from the right, if there's a discontinuity, $f'(a)$ **does not exist**!

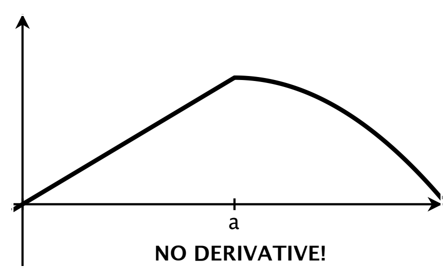


When can we take derivatives?

Not all functions have derivatives at all places. Before calculating $f'(a)$, first ask ...

(3) Is there a "corner" at $x = a$?

Next we'll explore how to find these algebraically, but if there's a sharp corner at $x = a$, then $f'(a)$ **does not exist**!



(Try drawing just one line that is tangent to that corner)

What's wrong with corners?

You try:

$$\text{Let } f(x) = \begin{cases} x^2 & x < 2, \\ x+2 & x \geq 2. \end{cases}$$

(a) Verify that $f(x)$ is continuous at $x = 2$.
(Compute $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $f(2)$, and compare.)

(b) Sketch a graph of $f(x)$.

(c) Estimate (ok to use a calculator), and then calculate the *right sided derivative*.

(d) Estimate (ok to use a calculator), and then calculate the *left sided derivative*.

(e) Compare your answers to (c) and (d), and explain why
$\lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$ does not exist.
Explain why $f'(2)$ does not exist.

Estimate the right-sided derivative: $f(2) = \boxed{}$

h	$f(2+h)$	$f(2+h)-f(2)$	$\frac{f(2+h)-f(2)}{h}$
1			
$1/2$			
$1/10$			

Compute the right-sided derivative:

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} =$$

Estimate the left-sided derivative: $f(2) = \boxed{}$

h	$f(2+h)$	$f(2+h)-f(2)$	$\frac{f(2+h)-f(2)}{h}$
-1			
$-1/2$			
$-1/10$			

Compute the right-sided derivative:

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} =$$