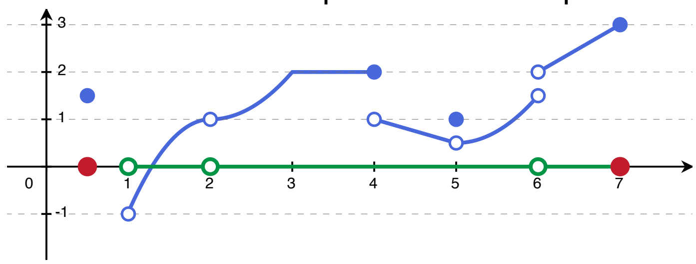
Recall: Continuity

Let a be an interior point or an endpoint of D.



Ex. $f(x)$ is discontinuous at $x = 4$ and 5.
No other points are fair game!

Let a be an interior point or an endpoint of D.
A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

Checklist:

1. Does (a) $\lim\limits_{x\to a^-} f(x)$ exist? (b) $\lim\limits_{x\to a^+} f(x)$ exist?
2. Does $\lim\limits_{x\to a} f(x)$ exist? (i.e. does (a) = (b)?)
3. Does $f(a) = \lim\limits_{x\to a} f(x)$?

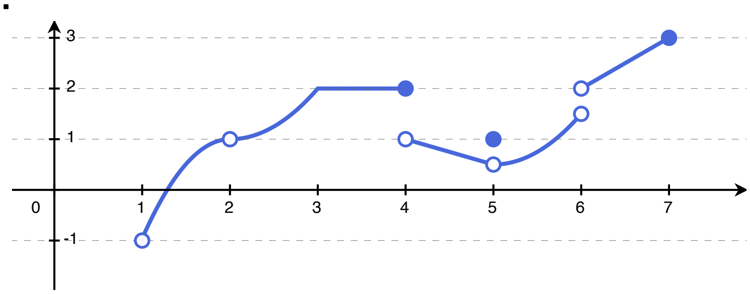
If the answer to any of 1.–3. is "no", then $f(x)$ is discontinuous at a.

Right Continuity and Left Continuity

Definition

A function $f(x)$ is right continuous at a point a if it is defined on an interval $[a, b)$ and $\lim_{x\to a^+} f(x) = f(a)$.
Similarly, a function $f(x)$ is left continuous at a point a if it is defined on an interval $(b, a]$ and $\lim_{x\to a^-} f(x) = f(a)$.

Example:



$f(x)$ is

(a) continuous at every *interior* point in D *except* $x = 4$ and 5;

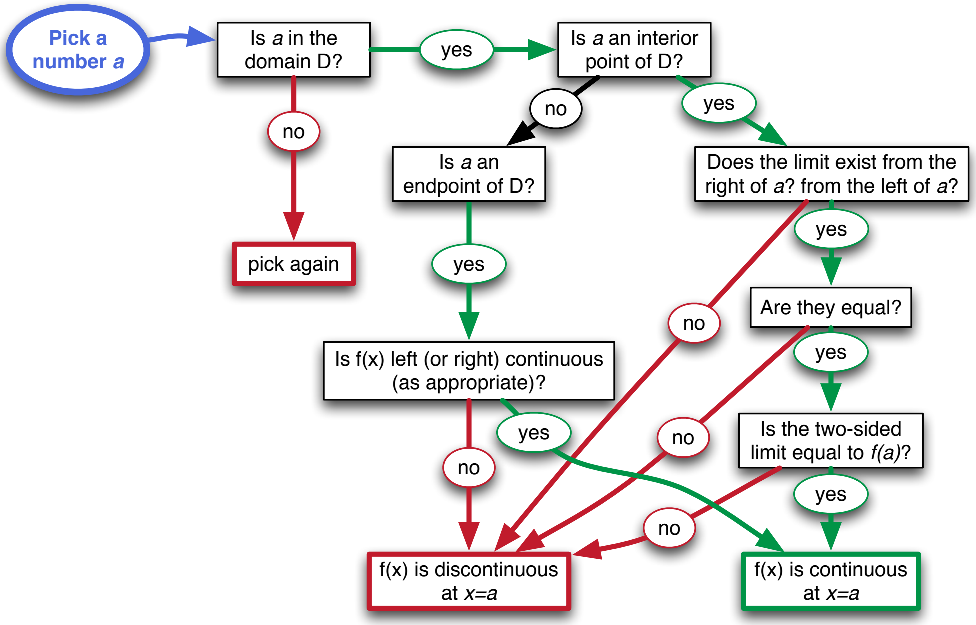
(b) only right continuous at those points included in (a); and

(c) additionally left continuous at $x = 4$ and $x = 7$.

Suppose a function f has no isolated points in its domain.

Definition

A function f is continuous over its domain D if **(1)** is is continuous at every interior point of D, and **(2)** it is left (or right) continuous at every endpoint of D. Otherwise, it has a discontinuity at each point in D which violates (1) or (2).



Filling and Fixing

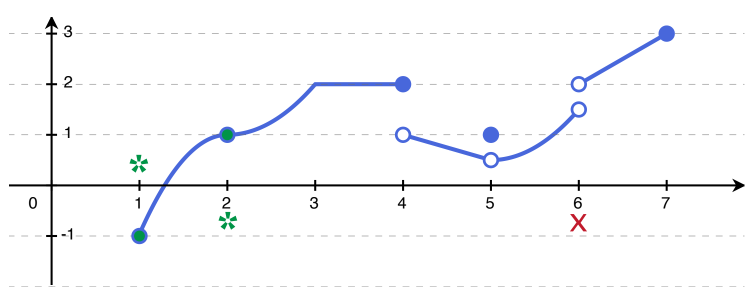
Suppose a is a hole in D (a is arbitrarily close to points in D, but not in D).

(a) If a would be an interior point and $\lim_{x\to a} f(x) = L$ exists; or

(b) if a would be an endpoint and $\lim_{x\to a^\pm} f(x) = L$ exists,

then we say $f(x)$ has a continuous extension:

$$\bar{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$$



Example: $f(x)$ has continuous extensions in exactly two places:

$$\bar{f}_1(x) = \begin{cases} f(x) & x \neq 1 \\ -1 & x = 1 \end{cases} \qquad \text{and} \qquad \bar{f}_2(x) = \begin{cases} f(x) & x \neq 2 \\ 1 & x = 2 \end{cases}$$

Examples

(A) Which of the following have removable discontinuities? For those which do, what are the alternate functions with those discontinuities removed?

(B) Which of the following have continuous extensions? For those which do, what are those extensions?

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$
2. $f(x) = \begin{cases} \sin x & x \neq \pi/3 \\ 0 & x = \pi/3 \end{cases}$
3. $f(x) = \dfrac{|x|}{x}$

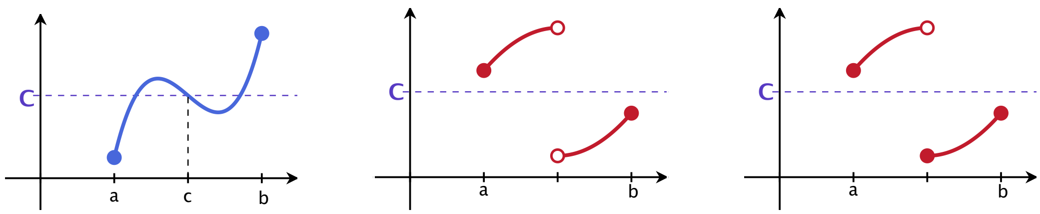
One application: The Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$.

If $\quad f(a) < C < f(b) \quad$ or $\quad f(a) > C > f(b),$

then there is at least one point c in the interval $[a, b]$ such that

$$f(c) = C.$$



Example 1: Show that the equation $x^5 - 3x + 1 = 0$ has at least one solution in the interval $[0, 1]$.
Example 2: Show every polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad a_n \neq 0$$

of odd degree has at least one real root (a solution to $p(x) = 0$).

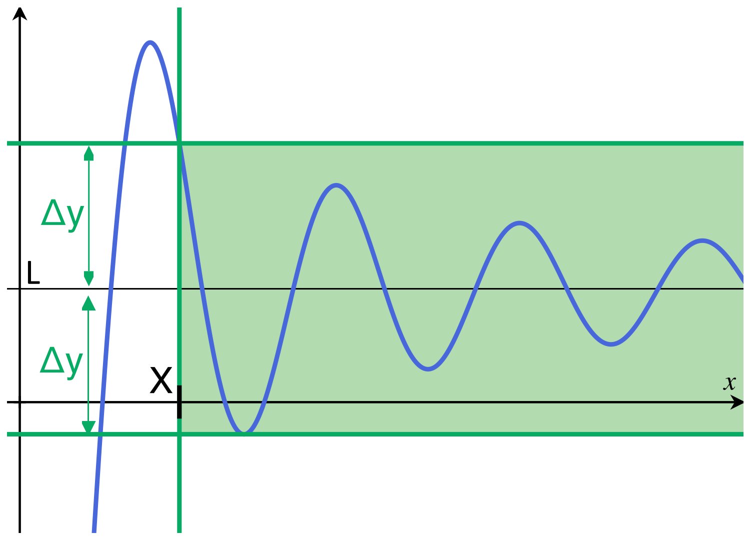
2.6 Limits involving infinity

Definition. We say that $f(x)$ has the limit L as x approaches infinity, written

$$\lim_{x \to \infty} f(x) = L$$

if for every $\Delta y > 0$ (think: smaller and smaller), there's some X for which

$$\text{whenever } x > X, \quad \text{we have } |f(x) - L| < \Delta y.$$



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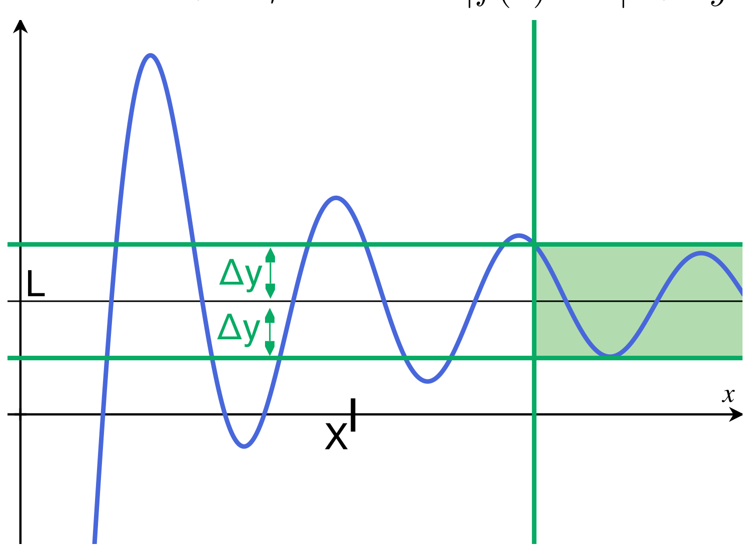

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if for every $\Delta y > 0$ (think: smaller and smaller), there's some X for which

$$\text{whenever } x > X, \quad \text{we have } |f(x) - L| < \Delta y.$$

"As x gets bigger and bigger, $f(x)$ stays closer and closer to L."

Similarly, we say $\lim_{x\to-\infty} f(x) = L$ if as x gets bigger and bigger in the negative directly, $f(x)$ stays closer and closer to L.

All limit rules from before, like sums, products, quotients, compositions, etc. all still apply.

All limit rules from before, like sums, products, quotients, compositions, etc. all still apply:

THEOREM 1—Limit Laws

If L, M, ~~c,~~ and k are real numbers and

$$\lim_{x\to\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to\infty}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\infty}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to\infty}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to\infty}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to\infty}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to\infty}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval ~~containing c~~ (a, ∞).)

(And similarly for $-\infty$)

Favorite examples:

$$\lim_{x\to\infty}\frac{1}{x}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x}=0$$

For any integer $n \geq 1$,

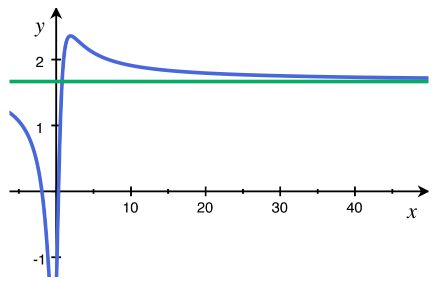
$$\lim_{x\to\infty}\frac{1}{x^n}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x^n}=0$$

Example:

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}\left(\frac{1/x^2}{1/x^2}\right)$$

$$=\lim_{x\to\infty}\frac{5(x^2/x^2)+8(x/x^2)-3/x^2}{3(x^2/x^2)+2/x^2}$$

$$=\lim_{x\to\infty}\frac{5+8(1/x)-3(1/x^2)}{3+2(1/x^2)}=\frac{5+8\cdot 0-3\cdot 0}{3+2\cdot 0}=\boxed{\frac{5}{3}}.$$



Favorite examples: For any integer $n \geq 1$,

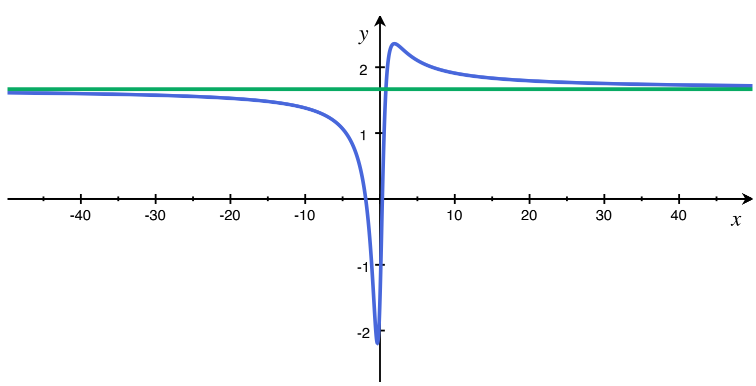
$$\lim_{x\to\infty}\frac{1}{x^n}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x^n}=0$$

Example: $\displaystyle\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\boxed{\frac{5}{3}}$.

Similarly,

$$\lim_{x\to-\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to-\infty}\frac{5x^2+8x-3}{3x^2+2}\left(\frac{1/x^2}{1/x^2}\right)$$

$$=\lim_{x\to-\infty}\frac{5+8(1/x)-3(1/x^2)}{3+2(1/x^2)}=\frac{5+8\cdot 0-3\cdot 0}{3+2\cdot 0}=\boxed{\frac{5}{3}}.$$



For any integer $n \geq 1$,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

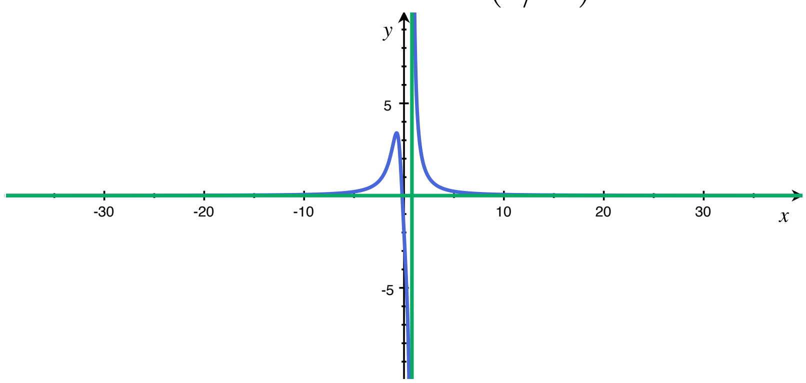
Example:

$$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{11x+2}{2x^3-1}\left(\frac{1/x^3}{1/x^3}\right) = \lim_{x\to\infty} \frac{11(x/x^3)+2(1/x^3)}{2(x^3/x^3)-(1/x^3)}$$

$$= \lim_{x\to\infty} \frac{11(1/x^2)+2(1/x^3)}{2-1/x^3} = \frac{11\cdot 0 + 2\cdot 0}{2-0} = \boxed{0}.$$

Similarly,

$$\lim_{x\to-\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to-\infty} \frac{11(1/x^2)+2(1/x^3)}{2-(1/x^3)} = \frac{0}{2} = \boxed{0}.$$



Last time: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$

Example. Let's compute $\lim_{x\to\infty} \sin(1/x)\cdot x$.

(Appears to approach "$0\cdot\infty$")

Make a substitution! Let $y = \frac{1}{x}$, so that $x = 1/y$. Also,

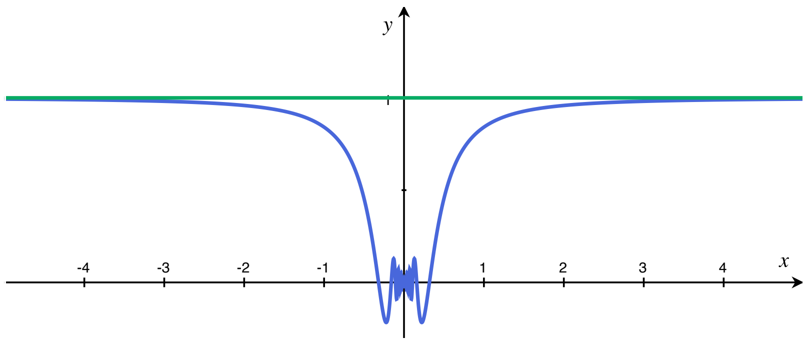
as $x \to \infty$, we have $y \to 0$.

So

$$\lim_{x\to\infty} \sin(1/x)\cdot x = \lim_{y\to 0} \sin(y)\cdot\frac{1}{y} = 1.$$

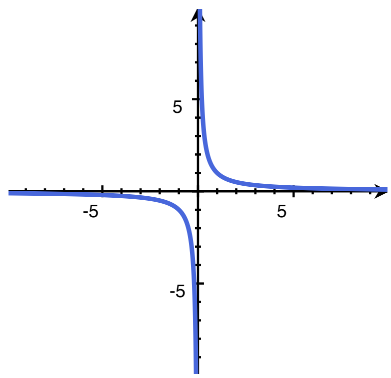
Similarly, as $x \to -\infty$, we have $y \to 0$. So

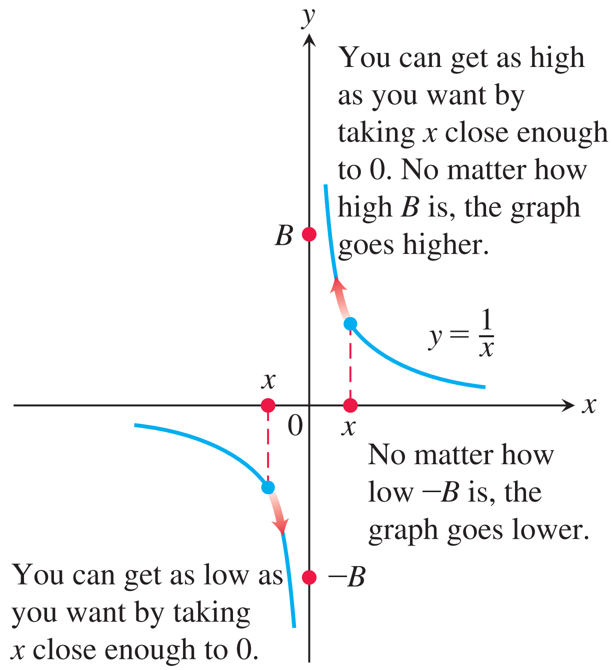
$$\lim_{x\to-\infty} \sin(1/x)\cdot x = \lim_{y\to 0} \sin(y)\cdot\frac{1}{y} = 1.$$



Infinite limits

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty$$





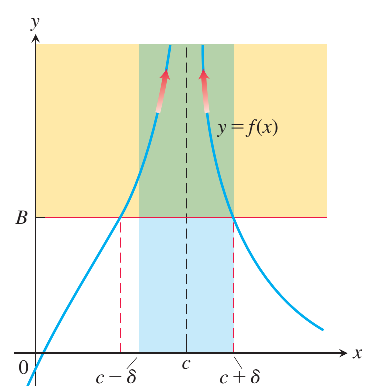
Formal definition:

We say that $f(x)$ **approaches infinity as x approaches c**, and write

$$\lim_{x \to c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$



General technique:

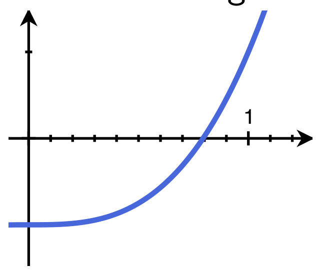
If $f(x) \to 0^{\pm}$ as $x \to c^{\pm}$,
then $1/f(x) \to \pm\infty$ as $x \to c^{\pm}$.
(Check signs one side at a time.)

Example: Compute $\lim_{x \to (\sqrt[3]{1/2})^-} \frac{11x+2}{2x^3-1}$ and $\lim_{x \to (\sqrt[3]{1/2})^-} \frac{11x+2}{2x^3-1}$.

As $x \to (\sqrt[3]{1/2})^-$, we have

$$11x + 2 \to 11\sqrt[3]{1/2} + 2 > 0 \quad \text{and} \quad 2x^3 - 1 \to 0^-.$$

$*$ Here, 0^- means 0 from the negative side, i.e. near $\sqrt[3]{1/2}$, but just to the left, we have $2x^3 - 1$ is negative.



So

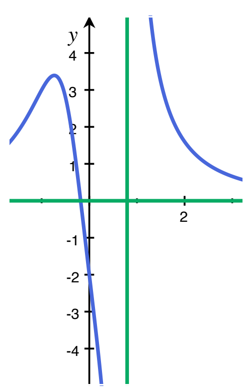
$$\boxed{\lim_{x \to (\sqrt[3]{2})^-} \frac{11x+2}{2x^3-1} = -\infty.} \quad \left(\frac{\text{pos}}{\text{neg}} = \text{neg}\right)$$

Similarly, As $x \to (\sqrt[3]{1/2})^+$, we have

$$11x + 2 \to 11\sqrt[3]{1/2} + 2 > 0 \quad \text{and} \quad 2x^3 - 1 \to 0^+.$$

Example: Compute $\lim_{x \to (\sqrt[3]{1/2})^-} \frac{11x+2}{2x^3-1}$ and $\lim_{x \to (\sqrt[3]{1/2})^-} \frac{11x+2}{2x^3-1}$.

$$\boxed{\lim_{x \to (\sqrt[3]{2})^-} \frac{11x+2}{2x^3-1} = -\infty.} \qquad \boxed{\lim_{x \to (\sqrt[3]{2})^+} \frac{11x+2}{2x^3-1} = \infty.}$$



Limits checklist

1. Can you just plug in? If so, do that.
2. Can you do some algebraic manipulation and cancel out problematic factors?
   (Most common: $f(x)/g(x)$ with $f(x) \to 0$ and $g(x) \to 0$.)
3. Do you know relevant special limits? (e.g. $\lim_{x\to 0} \sin(x)/x$)
4. Is your limit of the form $f(x)/g(x)$ with $g(x) \to 0$ and $f(x) \not\to 0$? Analyze one side at a time.

Infinite limits at infinity: long-term behavior

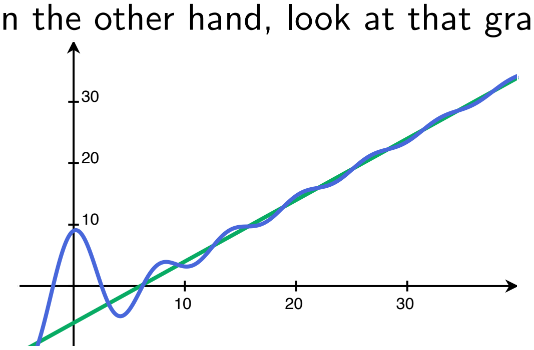
If $f(x) \to \infty$ as $x \to \infty$, sometimes we can do a better job of describing what's going on.

Namely, is the function growing slowly? exponentially? linearly? erratically? (Think: Zoom way out and look at the big picture. Does your function start to look like another simpler function after a while?)

Example: $\displaystyle\lim_{x\to\infty} 15\frac{\sin(x)}{x} + x - 6.$

Answer: On the one hand, we have $\frac{\sin(x)}{x} \to 0$ (by the sandwich theorem) and $x - 6 \to \infty$, so $\displaystyle\lim_{x\to\infty} 15\frac{\sin(x)}{x} + x - 6 = \infty.$

On the other hand, look at that graph!

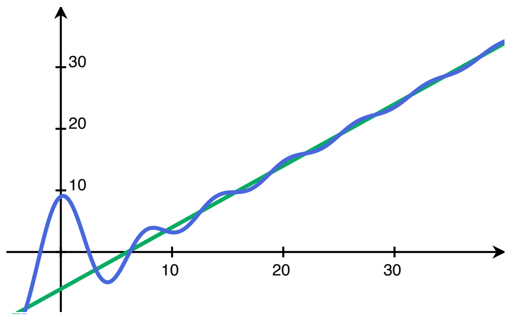


(As x gets large, $f(x)$ behaves more and more like the line $y = x - 6$.)

Long-term behavior: Suppose as $x \to \pm\infty$, we have

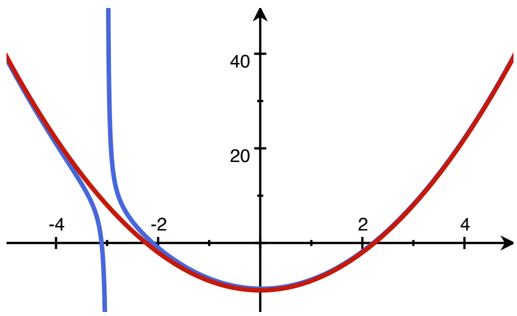
$$f(x) \to 0 \quad \text{and} \quad g(x) \not\to 0.$$

Then for "large x", we have $f(x) + g(x) \approx g(x)$.
We call $g(x)$ the dominant term(s).



Ex: For large x, we have

$$15\frac{\sin(x)}{x} + x - 6 \approx x - 6.$$



Ex: For large x, we have

$$\frac{1}{x+3} + 2x^2 - 10 \approx 2x^2 - 10.$$

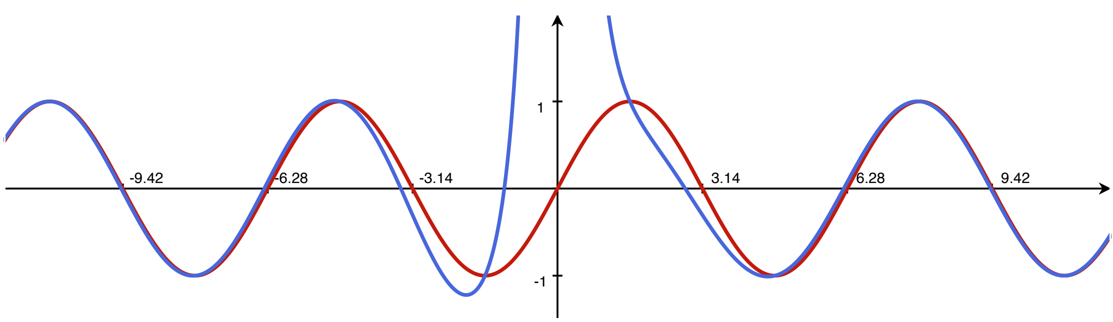
Long-term behavior: Suppose as $x \to \pm\infty$, we have

$$f(x) \to 0 \quad \text{and} \quad g(x) \not\to 0.$$

Then for "large x", we have $f(x) + g(x) \approx g(x)$.
We call $g(x)$ the dominant term(s).

Ex. For large x, we have

$$3\frac{\cos(x)}{x^2} + \sin(x) \approx \sin(x).$$



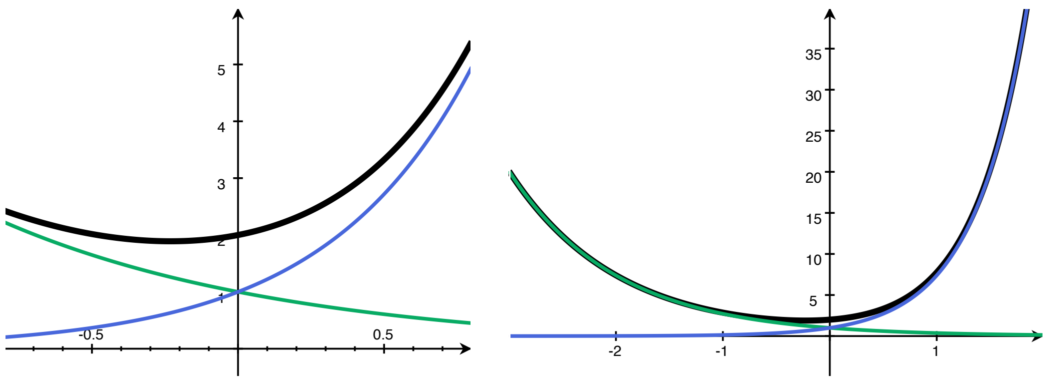
Long-term behavior: Suppose as $x \to \pm\infty$, we have

$$f(x) \to 0 \quad \text{and} \quad g(x) \not\to 0.$$

Then for "large x", we have $f(x) + g(x) \approx g(x)$.
We call $g(x)$ the dominant term(s).

Ex. For large **positive** x, we have $e^{-x} + e^{2x} \approx e^{2x}$,
and for large **negative** x, we have $e^{-x} + e^{2x} \approx e^{-x}$.

(So for large positive x, the function e^{2x} dominates;
and for large negative x, the function e^{-x} dominates.)



Disguised example: Note that

$$\lim_{x\to\infty} \frac{3-2x^2}{5x-1}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{3/x - 2x}{5 - 1/x} \begin{matrix} \to 0 - \infty \\ \to 5 - 0 \end{matrix} = -\infty.$$

But what general shape does $\frac{3-2x^2}{5x-1}$ take for large x?
Strategy: Put $p(x)/q(x)$ into the sum of polynomials and rational functions where each has deg(denominator) $>$ deg(numerator), via long division.

$$\begin{array}{r|cccccc} & -\frac{2}{5}x & - & \frac{2}{25} & + & \frac{(73/25)}{5x-1} \\ \hline 5x-1 & -2x^2 & + & 0\cdot x & + & 3 \\ & -(-2x^2 & + & \frac{2}{5}x) & & \\ \hline & 0 & + & -\frac{2}{5}x & + & 3 \\ & & & -(-\frac{2}{5}x & + & \frac{2}{25}) \\ \hline & & & & & \frac{73}{25} \end{array} \longleftarrow \deg(5x-1) > \deg(\tfrac{73}{25})\checkmark$$

So

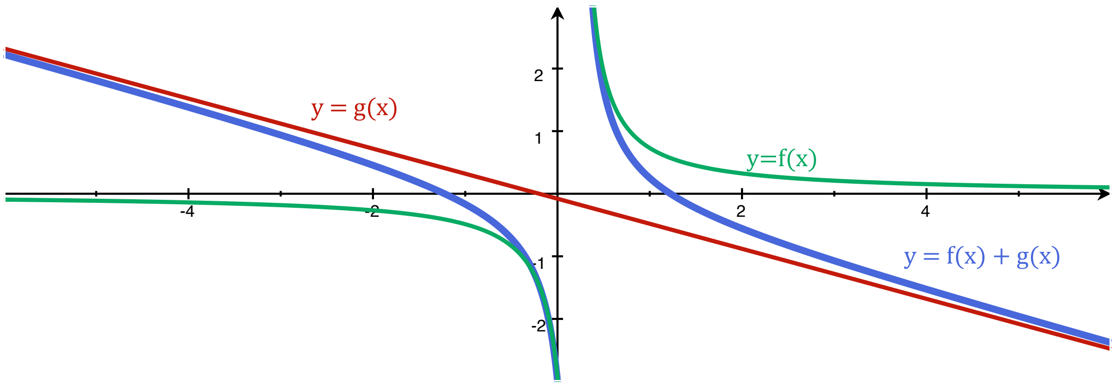
$$\boxed{\frac{3-2x^2}{5x-1} = -\frac{2}{5}x - \frac{2}{25} + \frac{(73/25)}{5x-1}.}$$

We have

$$\frac{3-2x^2}{5x-1} = \underbrace{-\frac{2}{5}x - \frac{2}{25}}_{g(x)} + \underbrace{\frac{(73/25)}{5x-1}}_{f(x)}.$$

And as $x \to \pm\infty$, we have $f(x) \to 0$ and $g(x) \not\to 0$. So for large x,

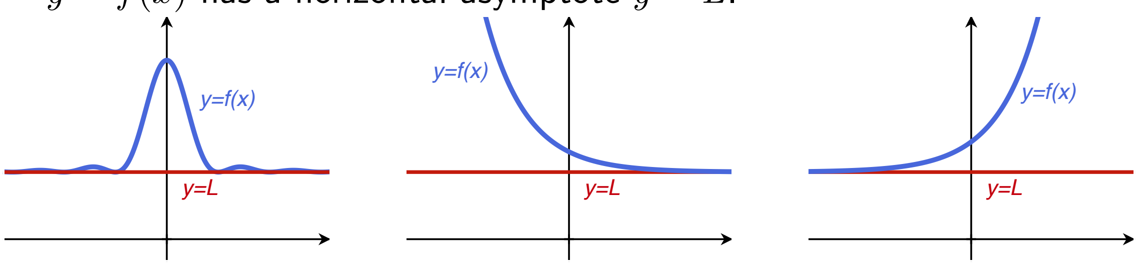
$$\frac{3-2x^2}{5x-1} \approx -\frac{2}{5}x - \frac{2}{25}.$$



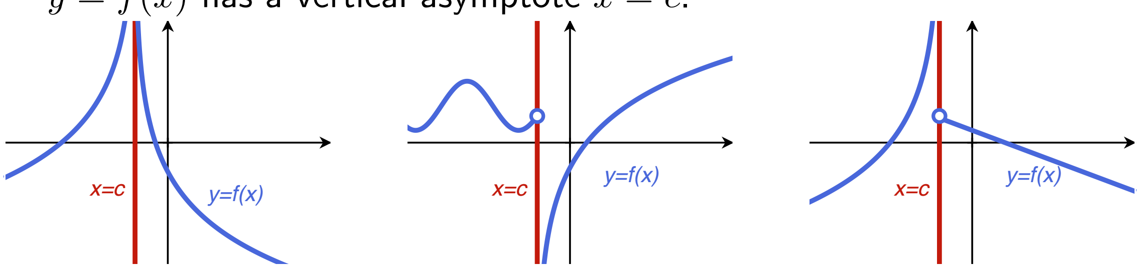
Of course, for x very close to $1/5$, $f(x)$ is **much** *smaller* than $g(x)$. So close to $x = 1/5$, we have $f(x) + g(x) \approx f(x)$, and $f(x)$ becomes the dominant term.

Limits and graphing: Asymptotes

Horizontal: If $f(x) \to L$ as $x \to \infty$ and/or as $x \to -\infty$, we say $y = f(x)$ has a horizontal asymptote $y = L$.



Vertical: If $f(x) \to \pm\infty$ as $x \to c^+$ and/or as $x \to c^-$, we say $y = f(x)$ has a vertical asymptote $x = c$.



Limits and graphing: Asymptotes

Oblique or **Slant line**: If $f(x) \approx mx + b$ for large (positive and/or negative) x, we say $y = f(x)$ has an oblique (a.k.a. slant line) asymptote $y = mx + b$.