Warmup.

Compute the following limits:

1. $\lim_{x \to 4} \dfrac{x^2 - 2x - 8}{x - 4}$;
2. $\lim_{x \to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$;
3. $\lim_{x \to 0} \dfrac{3 - \sqrt{9 - 2x}}{x}$.

Recall that a limit $\lim_{x \to a} f(x)$ exists whenever $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal. Let

$$f(x) = \begin{cases} 1/x & \text{for } x < -1, \\ -x^2 & \text{for } -1 \le x < 2, \\ 2x + 1 & \text{for } x \le 2, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \sin(x) & \text{for } x < \pi/2, \\ A & \text{for } x = \pi/2, \\ 2x + B & \text{for } \pi/2 < x. \end{cases}$$

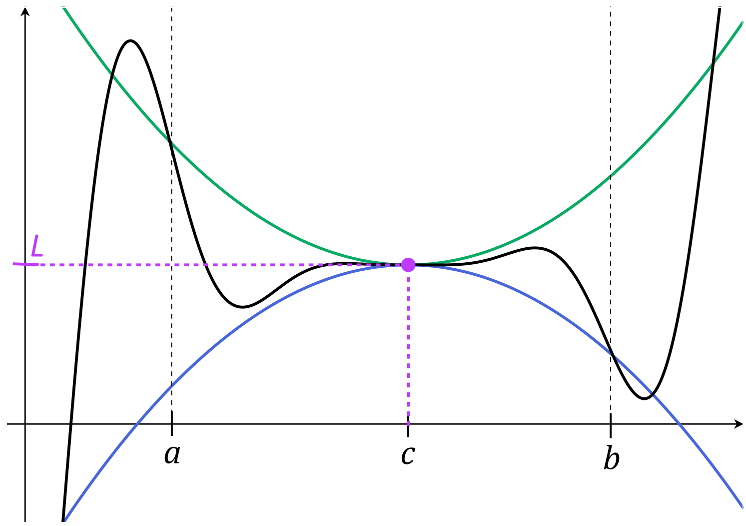
4. For which C does $\lim_{x \to C} f(x)$ exist?
5. For which A and B does $\lim_{x \to a} g(x)$ exist for all a?

Sandwich theorem

Fix $a \leq c \leq b$. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $a \leq x \leq b$ (except possibly for $x = c$). If

$$\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x),$$

then $\lim_{x \to c} f(x) = L$.



Sandwich theorem

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$$\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x),$$

then $\lim_{x \to c} f(x) = L$.

Example: Compute $\lim_{x \to 0} x^2 \sin(1/x)$.

Solution: Since

$$-1 \leq \sin(1/x) \leq 1 \quad \text{for all } x,$$

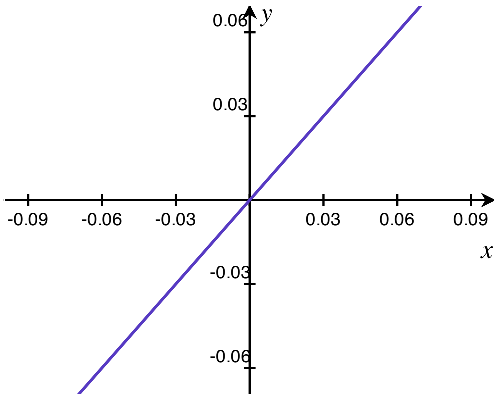
except at $x = 0$, where $\sin(1/x)$ is not defined. Then since $x^2 \geq 0$, we can multiply through by x^2 to get

$$-x^2 \leq x^2 \sin(1/x) \leq x^2 \quad \text{for all } x \neq 0.$$

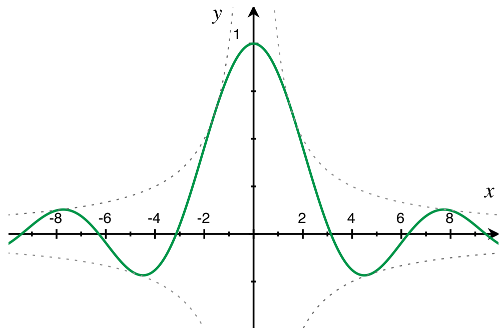
Further, $\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$. Thus $\boxed{\lim_{x \to 0} x^2 \sin(1/x) = 0}$.

One important limits

Near $x = 0$, $\sin(x) \approx x$:



Graph of $\frac{\sin(x)}{x}$:

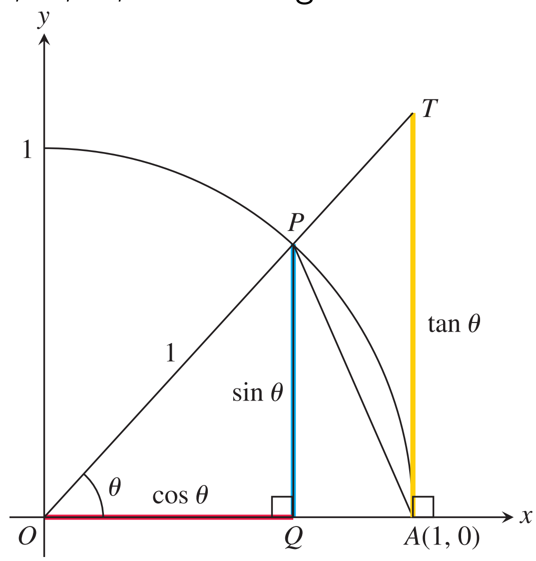


Hypothesis:

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Thm. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

Proof. Consider $0 < \theta < \pi/2$.
Let the points O, A, P, and T be given as follows:



Then

$$\text{Area}(\Delta OAP) \leq \text{Area}(\text{ wedge } OAP) \leq \text{Area}(\Delta OAT)\ldots$$

Thm. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1.$

Example. Compute $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x^2}$.

Solution. Recall

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad \text{and} \quad \cos^2(\theta) + \sin^2(\theta) = 1.$$

So considering $\theta = x/2$, we have

$$\begin{aligned}\cos(x) = \cos(2(x/2)) &= \cos^2(x/2) - \sin^2(x/2) \\ &= (1 - \sin^2(x/2)) - \sin^2(x/2) = 1 - 2\sin^2(x/2).\end{aligned}$$

So

$$\lim_{x\to 0} \frac{\cos(x)-1}{x^2} = \lim_{x\to 0} \frac{-2\sin^2(x/2)}{x^2} = -\left(\lim_{2\theta\to 0} \frac{\sin(\theta)}{\theta}\right)\left(\lim_{2\theta\to 0} \frac{\sin(\theta)}{\theta}\right).$$

Note as $x \to 0$, we have $\theta = x/2 \to 0$. So

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = -\left(\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\right)^2 = -(1)(1) = \boxed{-1}.$$

Thm. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1.$

Example. Compute $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{\sin(3x)}$.

Solution. We have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)} &= \lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)} \cdot \frac{3x}{2x} \cdot \frac{2}{3} \\ &= \frac{2}{3} \cdot \lim_{x\to 0} \frac{\sin(2x)/2x}{\sin(3x)/3x} = \frac{2}{3} \cdot \frac{\lim_{x\to 0} \sin(2x)/2x}{\lim_{x\to 0} \sin(3x)/3x}.\end{aligned}$$

As $x \to 0$, we have $2x \to 0$ and $3x \to 0$. Thus

$$\lim_{x\to 0} \sin(2x)/2x = \lim_{2x\to 0} \sin(2x)/2x = \lim_{y\to 0} \sin(y)/y = \boxed{1},$$

and similarly $\lim_{x\to 0} \sin(3x)/3x = 1$. Thus

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)} = \frac{2}{3} \cdot \frac{1}{1} = \boxed{2/3}.$$

Thm. $\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$

Example. Compute $\lim_{x\to 0} \frac{\sin(5x)}{x}$.

Solution. We have

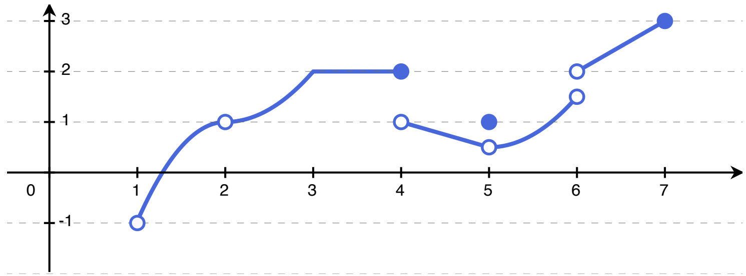
$$\lim_{x\to 0} \frac{\sin(5x)}{x} = \lim_{x\to 0} \frac{\sin(5x)}{5x} \cdot 5 = 5 \lim_{x\to 0} \frac{\sin(5x)}{5x}.$$

Again, as $x \to 0$, we have $5x \to 0$. Thus

$$\lim_{x\to 0} \frac{\sin(5x)}{x} = 5 \lim_{5x\to 0} \frac{\sin(5x)}{5x} = 5 \lim_{y\to 0} \frac{\sin(y)}{y} = 5 \cdot 1 = \boxed{5}.$$

Warm-up

Suppose the graph of $y = f(x)$ looks like

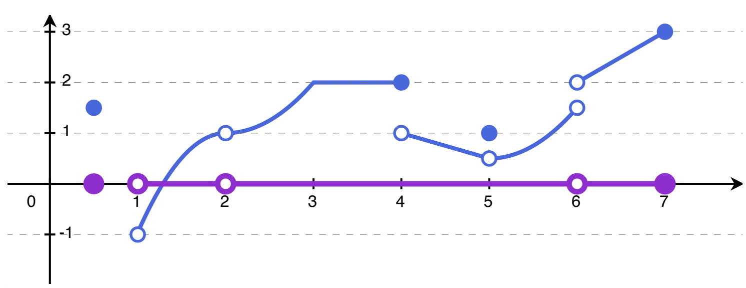


answers to 4.:

a	$x \to a^-$	$x \to a^+$

1. What is the domain of $f(x)$?
2. What is the range of $f(x)$?
3. For which values a in $[1, 7]$ does $\lim_{x \to a} f(x)$ not exist?
4. For those values you picked out in 3., what are
$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x)?$$
5. Which values a satisfy
$$f(a) \text{ and } \lim_{x \to a} f(x) \text{ exist, but } f(a) \neq \lim_{x \to a} f(x)?$$

Domain definitions



Let D be the domain of $f(x)$.

Ex. $D = (1, 2) \cup (2, 6) \cup (6, 7]$

Ex 2. $D = \{\frac{1}{2}\} \cup (1, 2) \cup (2, 6) \cup (6, 7]$

Definition

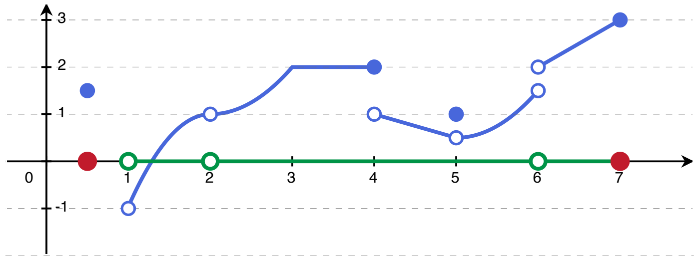
An interior point of D is any point **in** D which is not an endpoint or an isolated point.

Ex. Everything in D except $x = 7$.

Ex 2. Everything in D except $x = \frac{1}{2}$ & 7.

Continuity

Let a be an interior point or an endpoint of D.



Ex. $f(x)$ is discontinuous at $x = 4$ and 5. No other points are fair game!

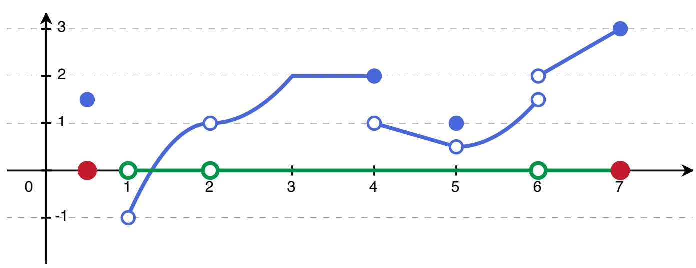
Definition

A function is

- right-continuous at a if $\lim_{x\to a^+} f(x) = f(a)$;
- left-continuous at a if $\lim_{x\to a^-} f(x) = f(a)$;
- continuous at a if $\lim_{x\to a} f(x) = f(a)$.

If a is an interior point and $f(x)$ it is not continuous at a, then function is discontinuous at a.

Continuity



Ex. $f(x)$ is discontinuous

at $x = 4$ and 5.

No other points are fair game!

Let a be an interior point. We say $f(x)$ is continuous at a if $\lim_{x\to a} f(x) = f(a)$. Otherwise, $f(x)$ is discontinuous at a.

Checklist:

1. Does (a) $\lim\limits_{x\to a^-} f(x)$ exist? (b) $\lim\limits_{x\to a^+} f(x)$ exist?
2. Does $\lim\limits_{x\to a} f(x)$ exist? (i.e. does (a) $=$ (b)?)
3. Does $f(a) = \lim\limits_{x\to a} f(x)$?

If the answer to any of 1.–3. is "no", then $f(x)$ is discontinuous at a.

Some examples:

Over their domains, all

polynomials, rational functions, trigonometric functions, exponential functions, absolute values,

and their inverses are all continuous functions.

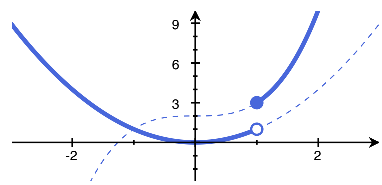
(Jumps all happen over domain gaps)

Example: Is the function $f(x) = \begin{cases} x^2 & x < 1 \\ x^3 + 2 & 1 \leq x \end{cases}$ continuous?

Solution: The only possible problem would happen at $x = 1$. Let's check there:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^3 + 2 = 3$$



No, $f(x)$ is discontinuous at $x = 1$ because 1 is an interior point of the domain, but $\lim_{x \to 1} f(x)$ does not exist.

Some examples:

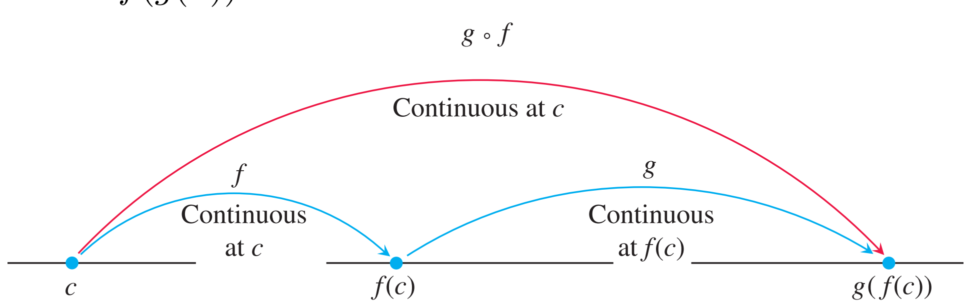
Over their domains, all

polynomials, rational functions, trigonometric functions, exponential functions, absolute values,

and their inverses are all continuous functions.

(Jumps all happen over domain gaps)

- Sums, differences, and products of continuous functions are continuous.
- If $g(c) \neq 0$ and $f(x)$ and $g(x)$ are continuous at c, then so is $g(x)/f(x)$.
- If $f(x)$ is continuous at c, and $g(x)$ is continuous at $f(c)$, then $f(g(x))$ is continuous at c.

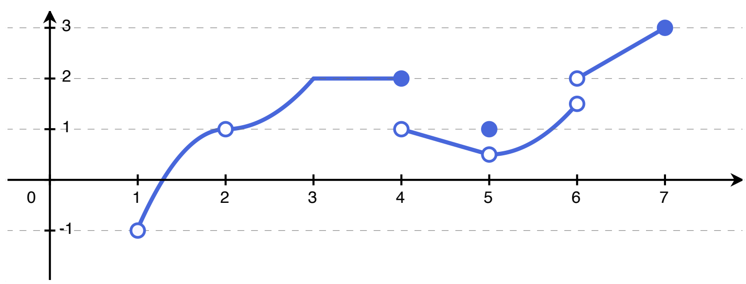


Right Continuity and Left Continuity

Definition

A function $f(x)$ is right continuous at a point a if it is defined on an interval $[a,b)$ and $\lim_{x\to a^+} f(x) = f(a)$.
Similarly, a function $f(x)$ is left continuous at a point a if it is defined on an interval $(b,a]$ and $\lim_{x\to a^-} f(x) = f(a)$.

Example:



$f(x)$ is

(a) continuous at every *interior* point in D *except* $x = 4$ and 5;

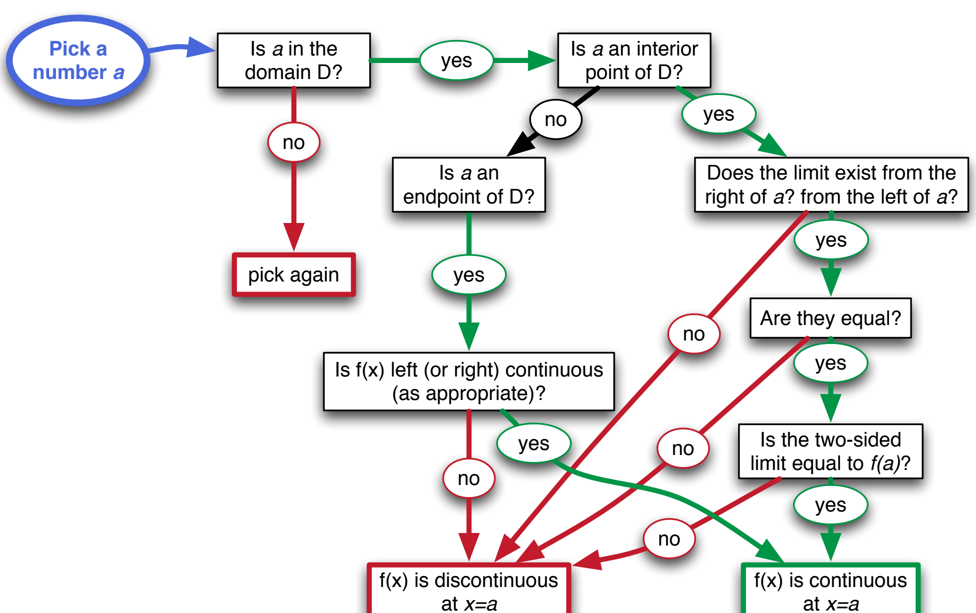
(b) only right continuous at those points included in (a); and

(c) additionally left continuous at $x = 4$ and $x = 7$.

Suppose a function f has no isolated points in its domain.

Definition

A function f is continuous over its domain D if **(1)** is is continuous at every interior point of D, and **(2)** it is left (or right) continuous at every endpoint of D. Otherwise, it has a discontinuity at each point in D which violates (1) or (2).



Filling and Fixing

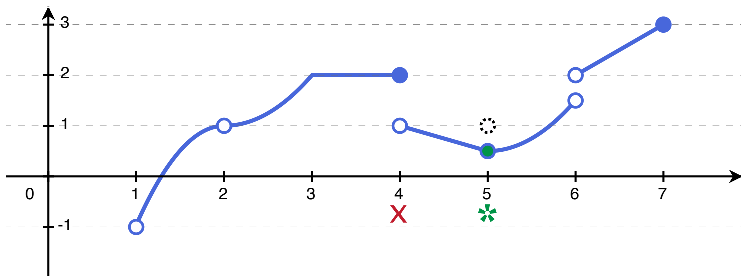
Suppose a is a point of discontinuity in D

(a) If a is an interior point and $\lim_{x\to a} f(x) = L$ exists; or

(b) if a is an endpoint and $\lim_{x\to a^{\pm}} f(x) = L$ exists,

then we say $f(x)$ has a removable discontinuity:

$$\bar{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$$



Example: $f(x)$ has a removable discontinuity in exactly one place:

$$\bar{f}(x) = \begin{cases} f(x) & x \neq 5 \\ 1/2 & x = 5 \end{cases}$$

Filling and Fixing

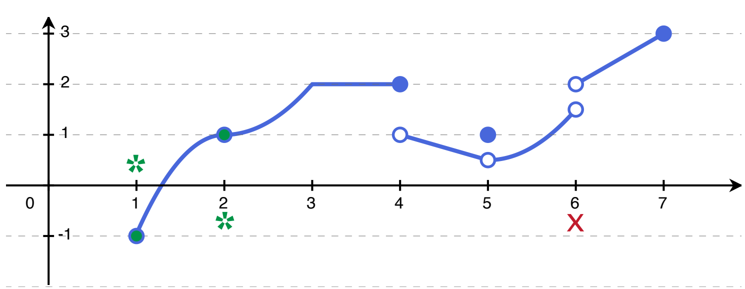
Suppose a is a hole in D (a is arbitrarily close to points in D, but not in D).

(a) If a would be an interior point and $\lim_{x\to a} f(x) = L$ exists; or

(b) if a would be an endpoint and $\lim_{x\to a^{\pm}} f(x) = L$ exists,

then we say $f(x)$ has a continuous extension:

$$\bar{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$$



Example: $f(x)$ has continuous extensions in exactly two places:

$$\bar{f}_1(x) = \begin{cases} f(x) & x \neq 1 \\ -1 & x = 1 \end{cases} \qquad \text{and} \qquad \bar{f}_2(x) = \begin{cases} f(x) & x \neq 2 \\ 1 & x = 2 \end{cases}$$

Examples

(A) Which of the following have removable discontinuities? For those which do, what are the alternate functions with those discontinuities removed?

(B) Which of the following have continuous extensions? For those which do, what are those extensions?

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$
2. $f(x) = \begin{cases} \sin x & x \neq \pi/3 \\ 0 & x = \pi/3 \end{cases}$
3. $f(x) = \dfrac{|x|}{x}$

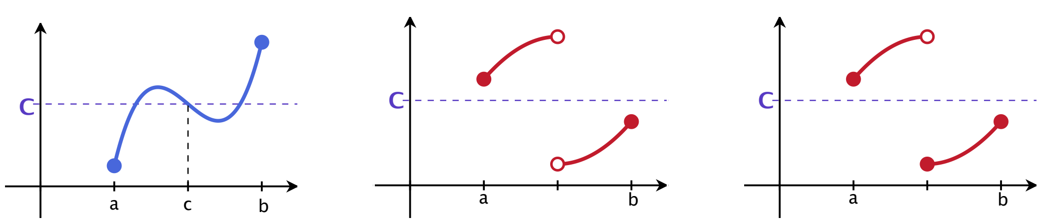
One application: The Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$.

If $\quad f(a) < C < f(b) \quad$ or $\quad f(a) > C > f(b),$

then there is at least one point c in the interval $[a, b]$ such that

$$f(c) = C.$$



Example 1: Show that the equation $x^5 - 3x + 1 = 0$ has at least one solution in the interval $[0, 1]$.
Example 2: Show every polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad a_n \neq 0$$

of odd degree has at least one real root (a solution to $p(x) = 0$).