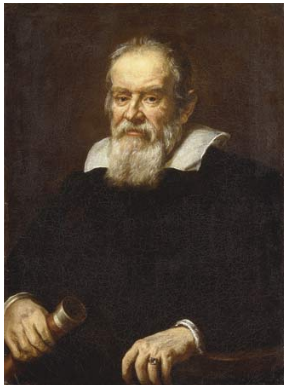
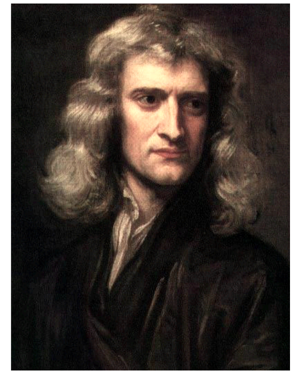
The Legacy of Galileo, Newton, and Leibniz



Galileo Galilei (1564-1642)
was interested in falling bodies.
He forged a new scientific methodology:

observe nature,
experiment to test what you observe, and construct theories that explain the observations.

Galileo (1564-1642): Experiment, then draw conclusions.



Sir Isaac Newton (1642-1727)
using his new tools of calculus, explained mathematically why an object, falling under the influence of gravity, will have constant acceleration of $9.8m/sec^2$.

His laws of motion unified
- Newton's laws of falling bodies,
- Kepler's laws of planetary motion,
- the motion of a simple pendulum,

and virtually every other instance of dynamic motion observed in the universe.

Galileo (1564-1642): Experiment, then draw conclusions.
Newton (1642-1727): Invented/used calculus to explain motion

Gottfried Wilhelm Leibniz (1646-1716)
independently co-invented calculus, taking a slightly different point of view ("infinitesimal calculus") but also studied rates of change in a general setting.

We take a lot of our notation from Leibniz.

Newton's Question:
How do we find the velocity of a moving object at time t?

What in fact do we mean by velocity of the object at the instant of time t? It's straightforward to find the average velocity of an object during a time interval $[t_1, t_2]$:

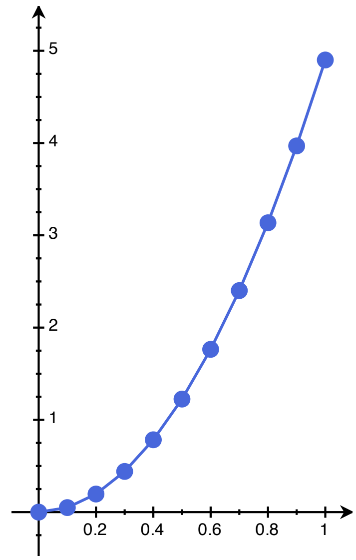
$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{\Delta y}{\Delta t}.$$

But what is meant by instantaneous velocity?

Drop a ball from the top of a building...

At time t, how far has the ball fallen? Measure it!

time (s)	distance (m)
0.10	0.049
0.20	0.196
0.30	0.441
0.40	0.784
0.50	1.225
0.60	1.764
0.70	2.401
0.80	3.136
0.90	3.969
1.00	4.900



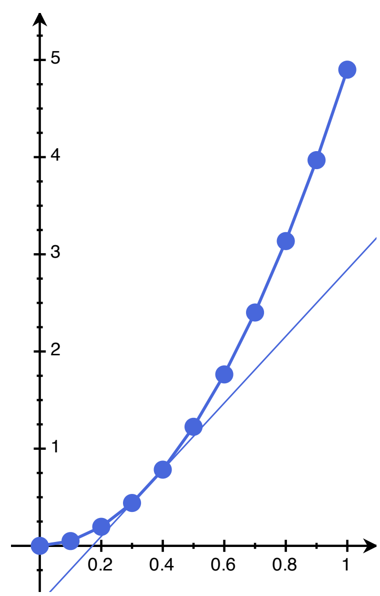
How fast is the ball falling at time t? A little trickier...

Average Speed

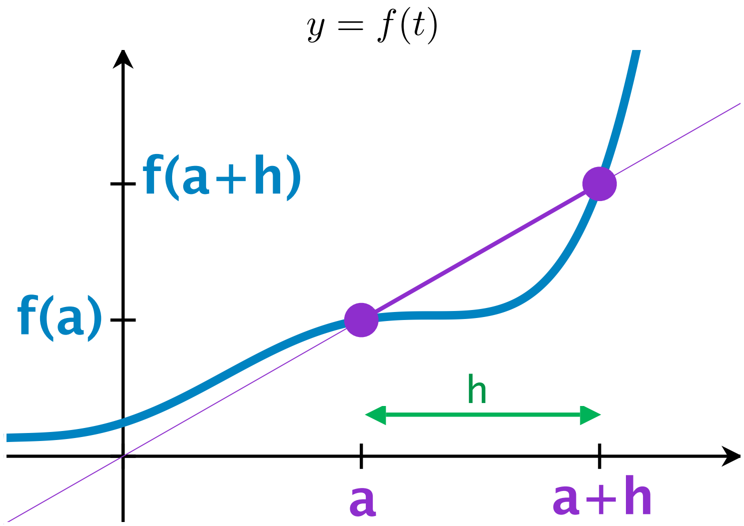
Definition

The **average velocity** from $t = t_1$ to $t = t_2$ is

$$\text{avg velocity} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \text{slope of secant line}$$



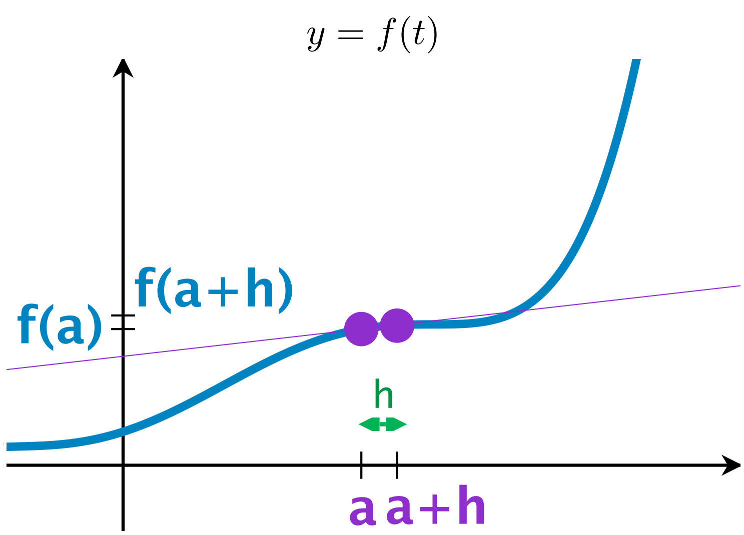
Plot position f versus time t:



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.
Slope of the line connecting them:

$$\text{avg velocity} = m = \frac{f(a+h) - f(a)}{h} \qquad \text{"difference quotient"}$$

Plot position f versus time t:



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.
Slope of the line connecting them:

$$\text{avg velocity} = m = \frac{f(a+h) - f(a)}{h} \qquad \text{"difference quotient"}$$

The smaller h is, the more useful m is!

Goal: Rates of change in general

Think: $f(x)$ is

distance versus time x, or
profit versus production volume x, or
birthrate versus population x, or...

Definition

Given a function f, the average rate of change of f over an interval $[x, x+h]$ is

$$\frac{f(x+h)-f(x)}{h}.$$

The average rate of change is also what we have called the difference quotient over the interval.

Definition

The instantaneous rate of change of a function at a point x is the limit of the average rates of change over intervals $[x, x+h]$ as $h \to 0$.

Future goals:

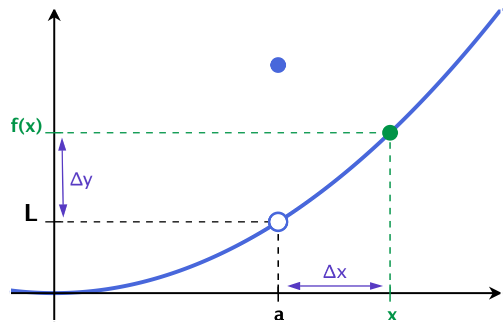
1. Get good at limits.
2. Explore instantaneous rates of change further, as limits of difference quotients.
3. Explore the geometric meaning of the definition of instantaneous rate of change at a point.
4. Apply the definition to each of the elementary functions to see if there are formula-like rules for calculating the instantaneous rate of change.
5. Use the definition of instantaneous rate of change and its consequences to obtain explicit functions for the position, velocity, and acceleration of a falling object.

Limit of a Function – Definition

We say that a function f approaches the limit L as x approaches a,

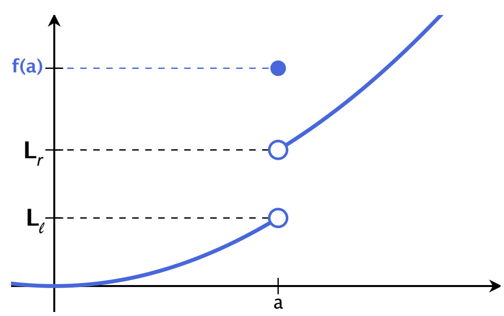
$$\text{written} \qquad \lim_{x \to a} f(x) = L,$$

if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a.



i.e. If you need Δy to be smaller,
you only need to make Δx smaller
(Δ means "change")

One-sided limits



Right-handed limit: $L_r = \lim_{x \to a^+} f(x)$

if $f(x)$ gets closer to L_r as x gets closer to a from the right

Left-handed limit: $L_\ell = \lim_{x \to a^-} f(x)$

if $f(x)$ gets closer to L_ℓ as x gets closer to a from the left

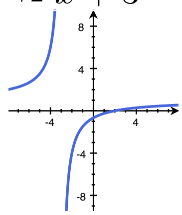
Theorem

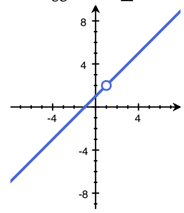
The limit of f as $x \to a$ exists if and only if both the right-hand and left-hand limits exist and have the same value, i.e.

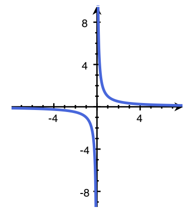
$$\lim_{x \to a} f(x) = L \textit{ if and only if } \lim_{x \to a^-} f(x) = L \textit{ and } \lim_{x \to a^+} f(x) = L.$$

Examples

$$\lim_{x\to 2}\frac{x-2}{x+3} \qquad \lim_{x\to 1}\frac{x^2-1}{x-1} \qquad \lim_{x\to 0}\frac{1}{x}$$





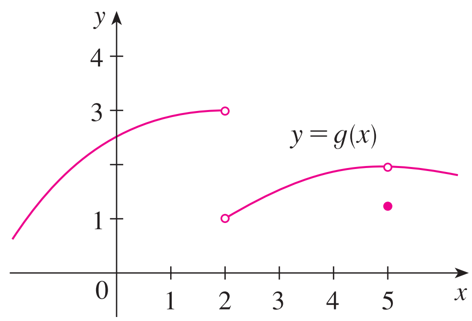


Compute

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

for the following function:



Theorem

If $\lim_{x \to a} f(x) = A$ *and* $\lim_{x \to a} g(x) = B$ *both exist, then*

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A + B$
2. $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A - B$
3. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$
4. If $B \neq 0$, then
   $\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x) = A/B$.

In short: to take a limit

Step 1: Can you just plug in? If so, do it.

Step 2: If not, is there some sort of algebraic manipulation (like cancellation) that can be done to fix the problem? If so, do it. Then plug in.

Step 3: Learn some special limit to fix common problems. (Later)

If in doubt, graph it!

Examples

1. $\lim_{x\to 2} \dfrac{x-2}{x+3} = \boxed{0}$ because if $f(x) = \frac{x-2}{x+3}$, then $f(2) = 0$.
2. $\lim_{x\to 1} \dfrac{x^2-1}{x-1} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$

   If $f(x) = \frac{x^2-1}{x-1}$, then $f(x)$ is undefined at $x = 1$.
   However, so long as $x \neq 1$,

   $$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1.$$

   So

   $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = \boxed{2}.$$

3. $\lim_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$, so again, $f(x)$ is undefined at a.

Examples

3. $\lim_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$, so again, $f(x)$ is undefined at a.

   Multiply top and bottom by the conjugate:

   $$\begin{aligned}
   \lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} &= \lim_{x\to 0} \left(\frac{\sqrt{x+2}-\sqrt{2}}{x}\right)\left(\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right) \\
   &= \lim_{x\to 0} \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})} \qquad \text{since } (a-b)(a+b) = a^2-b^2 \\
   &= \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})} \\
   &= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}
   \end{aligned}$$

You try:

1. $\lim_{x\to 1} \frac{x^2-3x+2}{x^2+4x-5}$

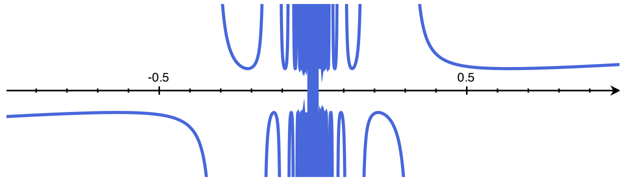
2. $\lim_{x\to -2} \frac{|x|}{x}$

3. $\lim_{x\to 0} \frac{|x|}{x}$

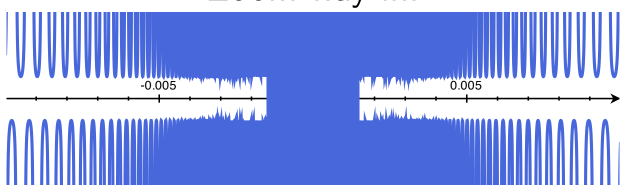
4. $\displaystyle\lim_{x\to 0} \frac{(3+x)^2-3^2}{x}$

Badly behaved example:

$$f(x) = \csc(1/x)$$



Zoom way in:



(denser and denser vertical asymptotes)

$\lim_{x\to 0^+} \csc(1/x)$ does not exist, and $\lim_{x\to 0^-} \csc(1/x)$ does not exist