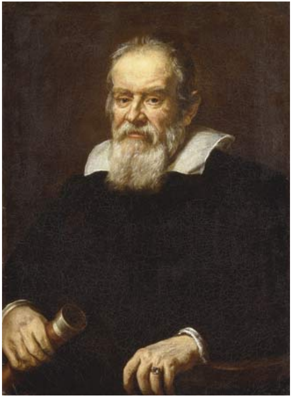
Modeling Rates of Change: Introduction to the Issues

The Legacy of Galileo, Newton, and Leibniz

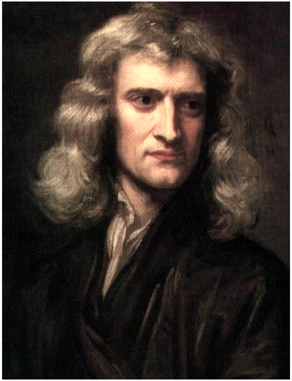


Galileo Galilei (1564-1642)
was interested in falling bodies.
He forged a new scientific methodology:

observe nature,
experiment to test what you observe, and
construct theories that explain the
observations.

The Legacy of Galileo, Newton, and Leibniz

Galileo (1564-1642): Experiment, then draw conclusions.



Sir Isaac Newton (1642-1727)
using his new tools of calculus, explained mathematically why an object, falling under the influence of gravity, will have constant acceleration of $9.8 m/sec^2$.

His laws of motion unified

Newton's laws of falling bodies,

Kepler's laws of planetary motion,

the motion of a simple pendulum,

and virtually every other instance of dynamic motion observed in the universe.

The Legacy of Galileo, Newton, and Leibniz

Galileo (1564-1642): Experiment, then draw conclusions.
Newton (1642-1727): Invented/used calculus to explain motion

Gottfried Wilhelm Leibniz (1646-1716) independently co-invented calculus, taking a slightly different point of view ("infinitesimal calculus") but also studied rates of change in a general setting.

We take a lot of our notation from Leibniz.

The Legacy of Galileo, Newton, and Leibniz

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What in fact do we mean by velocity of the object at the instant of time t?

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But what is meant by instantaneous velocity?

Drop a ball from the top of a building...

At time t, how far has the ball fallen?

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At time t, how far has the ball fallen? Measure it!

time (s)	distance (m)
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0.30	0.441
0.40	0.784
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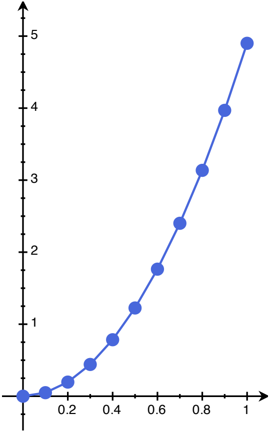

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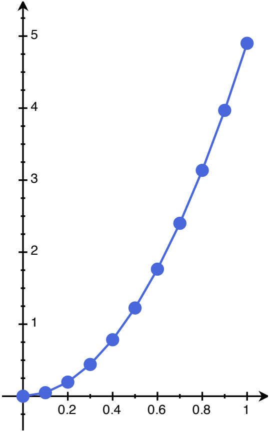

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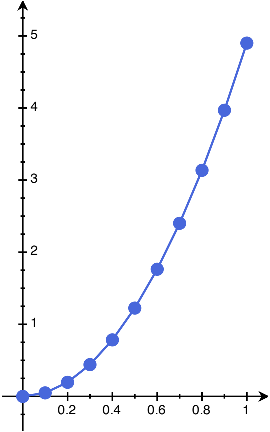

How fast is the ball falling at time t? A little trickier...

Average Speed

Definition

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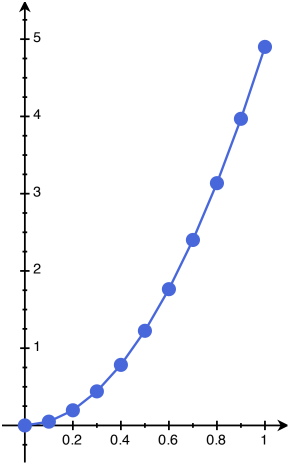

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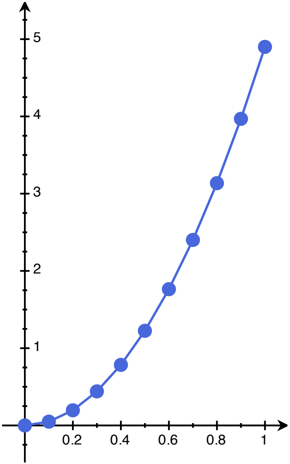

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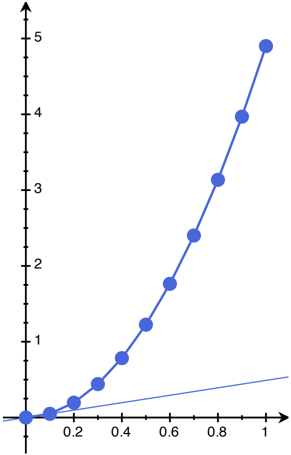

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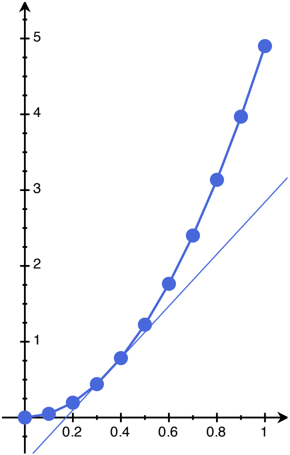

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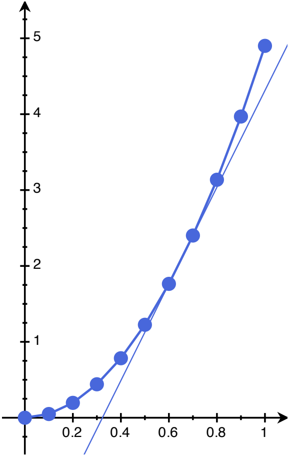

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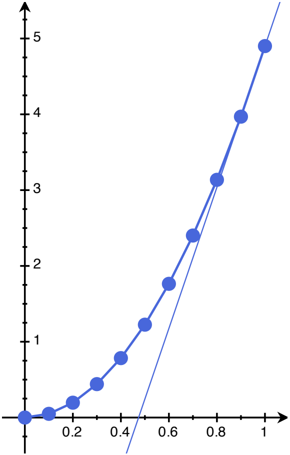
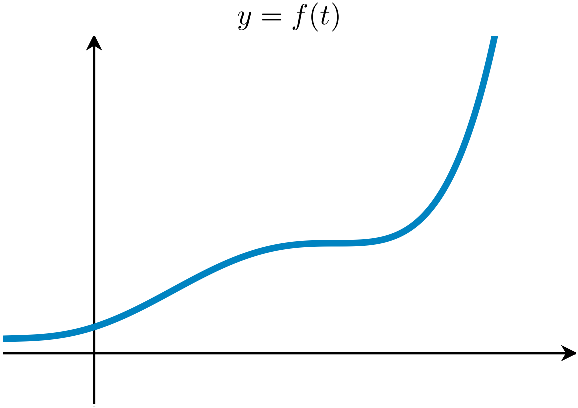


Plot position f versus time t:



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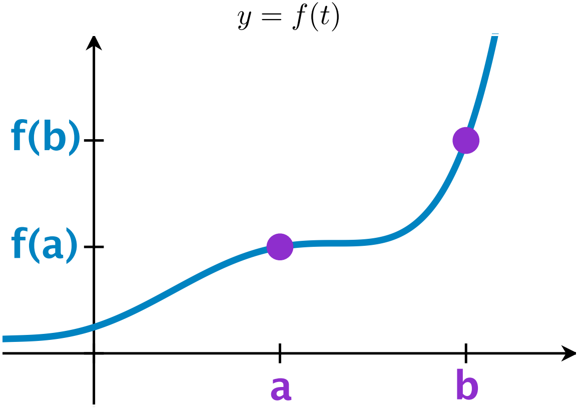
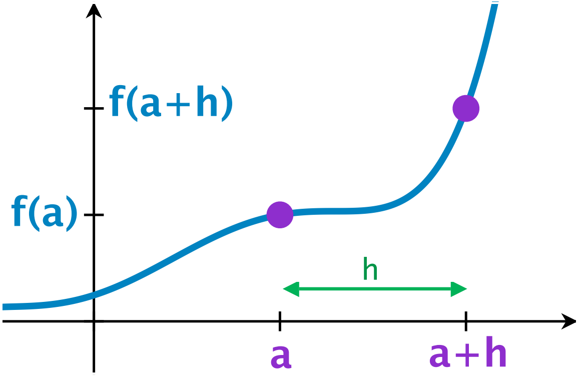


Pick two points on the curve $(a, f(a))$ and $(b, f(b))$.

Plot position f versus time t:

$y = f(t)$



Pick two points on the curve $(a, f(a))$ and $(b, f(b))$. Rewrite $b = a + h$.

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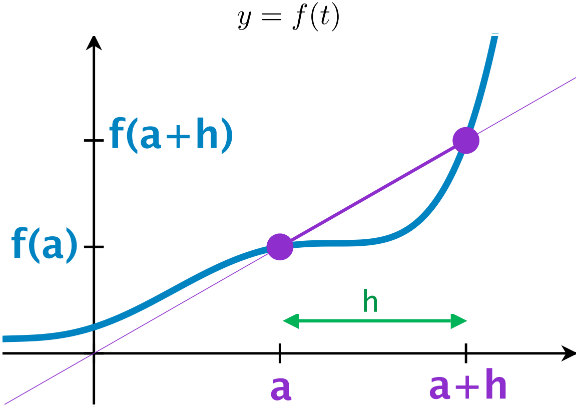

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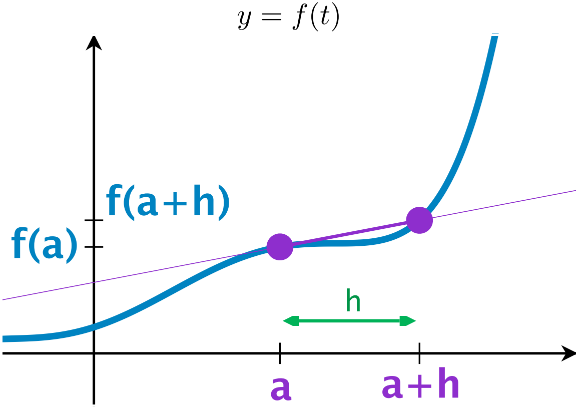

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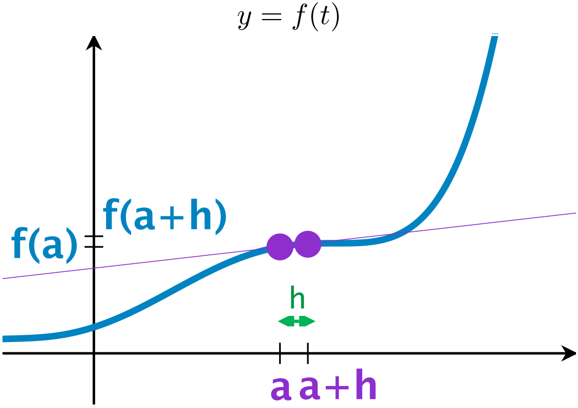

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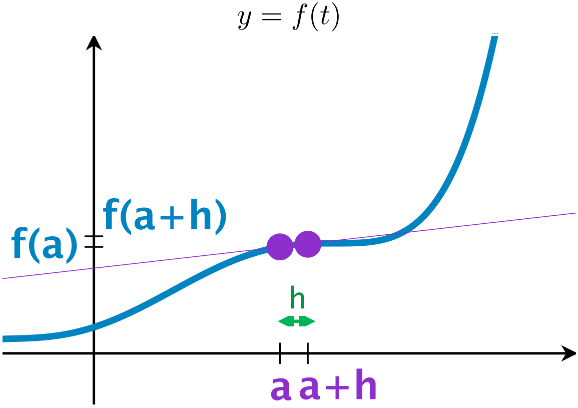

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The smaller h is, the more useful m is!

Goal: Rates of change in general

Think: $f(x)$ is

distance versus time x, or
profit versus production volume x, or
birthrate versus population x, or...

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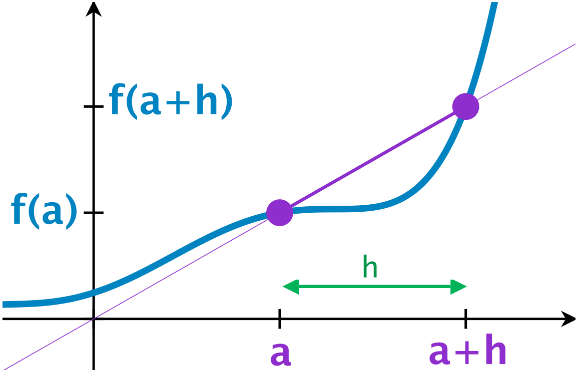
Definition

The instantaneous rate of change of a function at a point x is the limit of the average rates of change over intervals $[x, x+h]$ as $h \to 0$.

Average rate of change → Instantaneous rate of change



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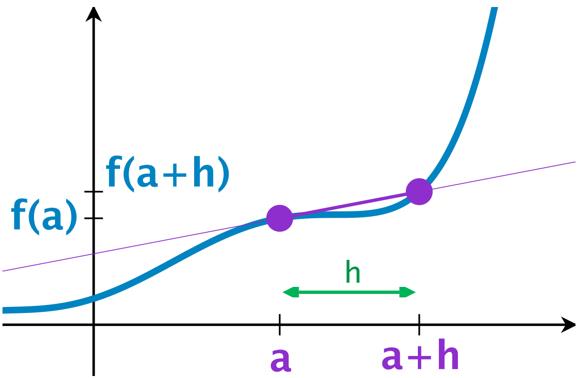

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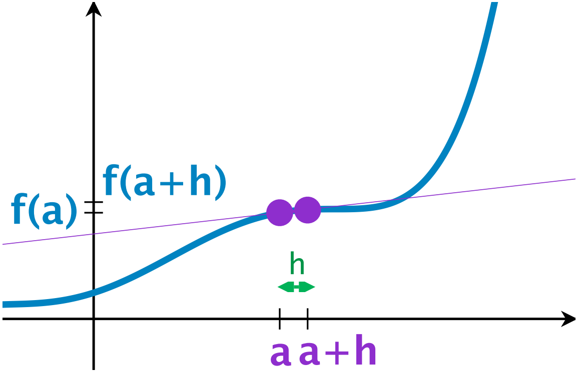

Future goals:

1. Get good at limits.
2. Explore instantaneous rates of change further, as limits of difference quotients.
3. Explore the geometric meaning of the definition of instantaneous rate of change at a point.
4. Apply the definition to each of the elementary functions to see if there are formula-like rules for calculating the instantaneous rate of change.
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Limit of a Function – Definition

We say that a function f approaches the limit L as x approaches a,

$$\text{written} \qquad \lim_{x \to a} f(x) = L,$$

if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a.

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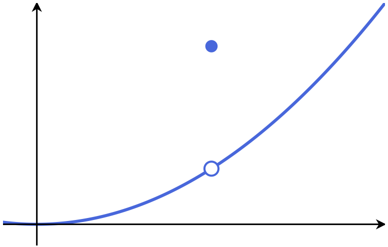

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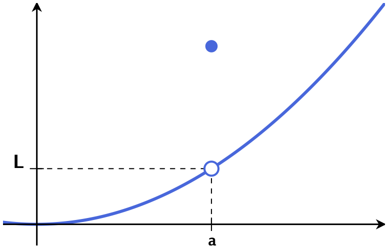

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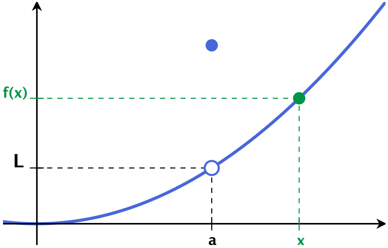

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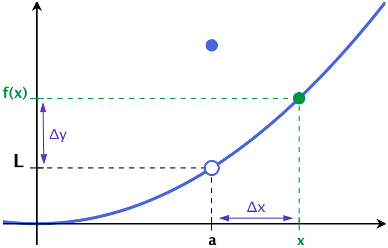

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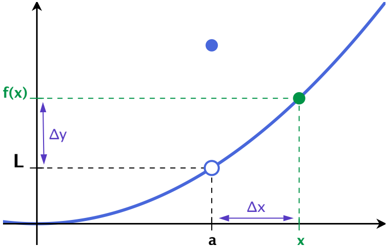
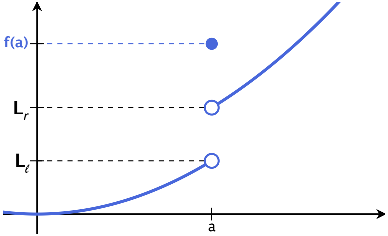


i.e. If you need Δy to be smaller,
you only need to make Δx smaller
(Δ means “change”)

One-sided limits



Right-handed limit: $L_r = \lim_{x \to a^+} f(x)$

if $f(x)$ gets closer to L_r as x gets closer to a from the right

Left-handed limit: $L_\ell = \lim_{x \to a^-} f(x)$

if $f(x)$ gets closer to L_ℓ as x gets closer to a from the left

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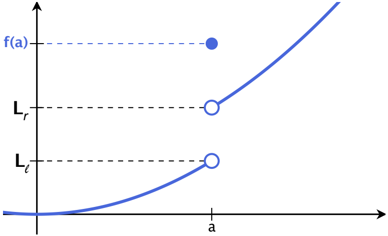

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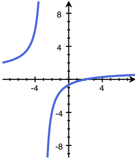
Theorem

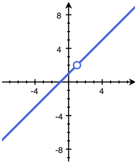
The limit of f as $x \to a$ exists if and only if both the right-hand and left-hand limits exist and have the same value, i.e.

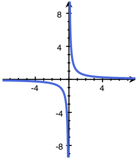
$$\lim_{x \to a} f(x) = L \textit{ if and only if } \lim_{x \to a^-} f(x) = L \textit{ and } \lim_{x \to a^+} f(x) = L.$$

Examples

$$\lim_{x\to 2} \frac{x-2}{x+3} \qquad \lim_{x\to 1} \frac{x^2-1}{x-1} \qquad \lim_{x\to 0} \frac{1}{x}$$





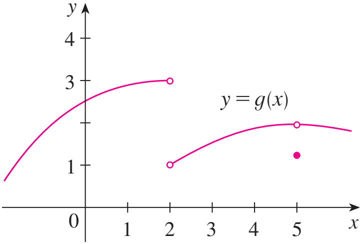


Compute

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

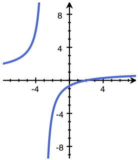
(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

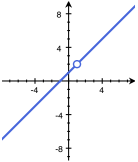
for the following function:

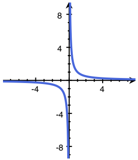


Examples

$$\lim_{x\to 2}\frac{x-2}{x+3}=0 \qquad \lim_{x\to 1}\frac{x^2-1}{x-1}=2 \qquad \lim_{x\to 0}\frac{1}{x} \text{ is undefined}$$







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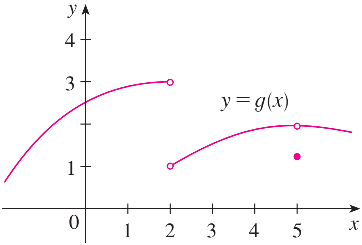

Theorem

If $\lim_{x\to a} f(x) = A$ *and* $\lim_{x\to a} g(x) = B$ *both exist, then*

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A + B$
2. $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A - B$
3. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$
4. If $B \neq 0$, then $\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x) = A/B$.

In short: to take a limit

Step 1: Can you just plug in? If so, do it.

Step 2: If not, is there some sort of algebraic manipulation (like cancellation) that can be done to fix the problem? If so, do it. Then plug in.

Step 3: Learn some special limit to fix common problems. (Later)

If in doubt, graph it!

Examples

1. $\lim_{x \to 2} \frac{x-2}{x+3}$
2. $\lim_{x \to 1} \frac{x^2-1}{x-1}$
3. $\lim_{x \to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$

Examples

1. $\lim_{x \to 2} \frac{x-2}{x+3} = \boxed{0}$ because if $f(x) = \frac{x-2}{x+3}$, then $f(2) = 0$.
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2. $\lim_{x \to 1} \frac{x^2-1}{x-1} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$
3. $\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

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   If $f(x) = \frac{x^2-1}{x-1}$, then $f(x)$ is undefined at $x = 1$.

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   However, so long as $x \neq 1$,

   $$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1.$$

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&= \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}}
\end{aligned}$$

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&= \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}
\end{aligned}$$

You try:

1. $\lim_{x\to 1} \frac{x^2-3x+2}{x^2+4x-5}$

2. $\lim_{x\to -2} \frac{|x|}{x}$

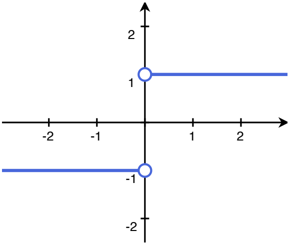
3. $\lim_{x\to 0} \frac{|x|}{x}$

4. $$\lim_{x\to 0} \frac{(3+x)^2 - 3^2}{x}$$

You try:

1. $\lim_{x\to 1} \frac{x^2-3x+2}{x^2+4x-5}$ $= \boxed{-\frac{1}{6}}$

2. $\lim_{x\to -2} \frac{|x|}{x} = \boxed{-1}$

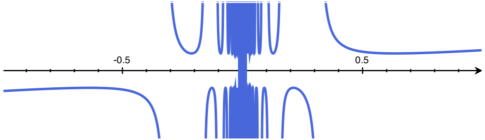


3. $\lim_{x\to 0} \frac{|x|}{x}$ is $\boxed{\text{undefined}}$

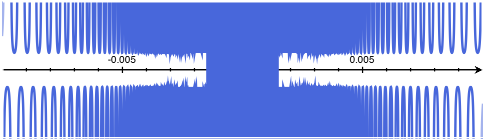
4. $\displaystyle\lim_{x\to 0} \frac{(3+x)^2 - 3^2}{x}$ $= \boxed{6}$

Badly behaved example:

$f(x) = \csc(1/x)$



Zoom way in:



(denser and denser vertical asymptotes)

$\lim_{x \to 0^+} \csc(1/x)$ does not exist, and $\lim_{x \to 0^-} \csc(1/x)$ does not exist