Inverse functions and logarithms

Recall that a function is a machine that takes a number from one set and puts a number of another set. Must be well-defined, meaning the function is decisive: (1) always has an answer and (2) always puts out one answer for each number taken in.

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$; e.g.

1	-2	$-\pi$	$1/3$	etc.
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
1	4	π^2	$1/9$	

2. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto |\sqrt{x}|$; e.g.

1	4	π^2	$1/9$	etc.
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
1	2	π	$1/3$	

Note that $\sqrt{x}$ is only a function when we go to extra effort to decide that we're always going to choose the positive answer.

3. Let bacteria grow, and measure population over time. Consider $N : \mathbb{N} \to \mathbb{N}$ by $N(t) = \#$ bacteria at time t.

t (hours)	$N(t) =$ pop. at time t
0	100
1	168
2	259
3	258
4	445
5	509

Now suppose we we're trying to ask the question "how long will it take to grow at least 500 bacteria?"

Answer: between 4 and 5 hours

Inverse functions

Given a function f, the inverse function f^{-1} is the machine that takes in f's output, and returns the corresponding input.

$$x \overset{f}{\longmapsto} f(x) \overset{f^{-1}}{\longmapsto} x$$

In notation, we write that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x.$$

Example: If $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by

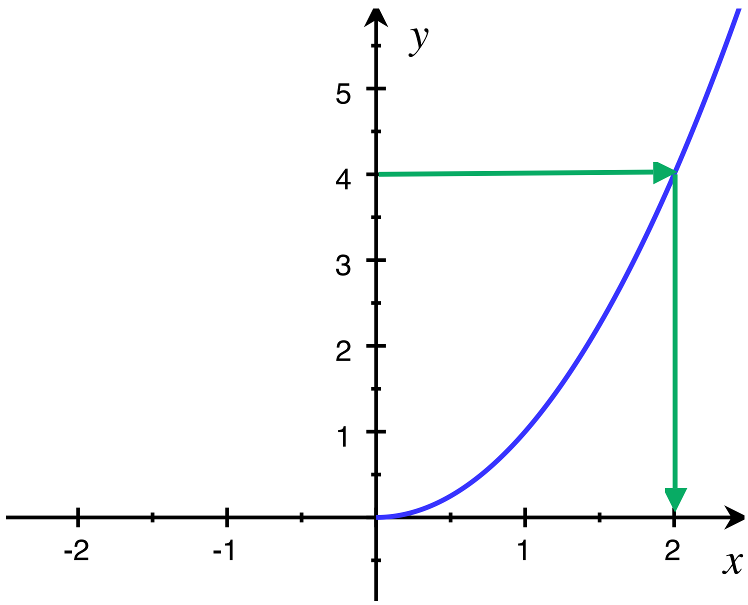
$$f(x) = x^2, \qquad \text{then} \qquad f^{-1}(x) = |\sqrt{x}| = \sqrt{x}.$$

Non-Example: If $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is given by

$$f(x) = x^2, \qquad \text{then} \qquad f^{-1}(x) \text{ is not well-defined}.$$

If $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by

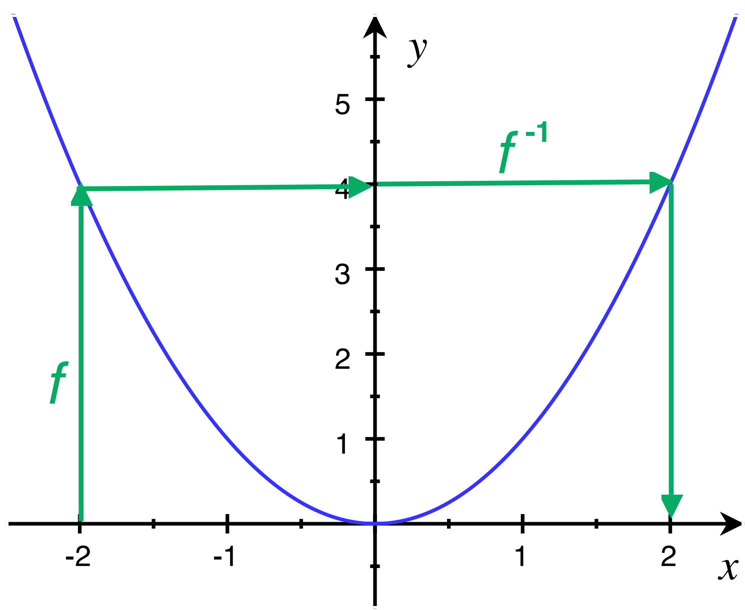
$$f(x) = x^2, \qquad \text{then} \qquad f^{-1}(x) = |\sqrt{x}|.$$



If $y = x^2$ and $x \geq 0$, then $x = |\sqrt{y}|$.

If $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is given by

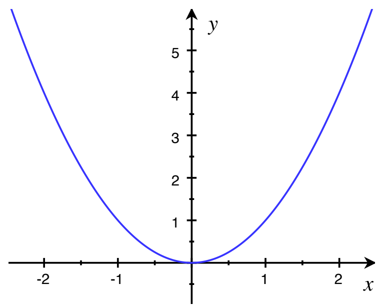
$$f(x) = x^2, \qquad \text{then} \qquad |\sqrt{x}| \text{ is not the inverse}.$$



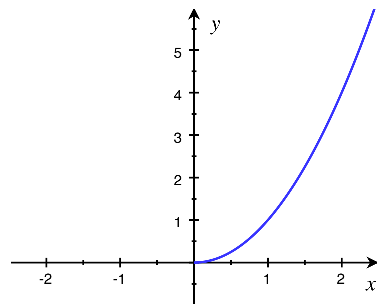
If $y = x^2$ and $x < 0$, then $x \neq |\sqrt{y}|$!

When is a function invertible?

A function f is one-to-one if no two inputs give the same output, that is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
Example: over all real numbers, $f(x) = x^2$ is not one-to-one.
However, over non-negative real numbers, $f(x) = x^2$ is one-to-one.



no!



yes!

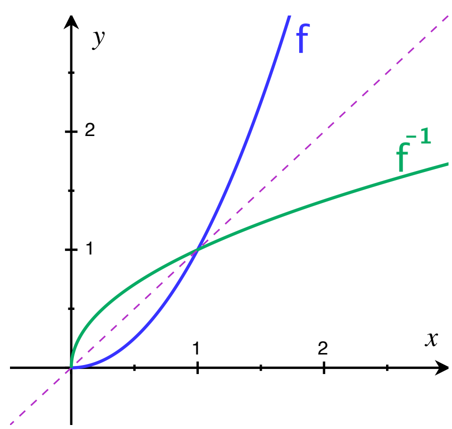
Horizontal line test: A function is one-to-one if and only if no horizontal line intersects the function's graph more than once.

Answer: A function is invertible if and only if it is one-to-one.

Graphing inverses

For a one-to-one function f, we have

$$f(x) = y \qquad \text{if and only if} \qquad x = f^{-1}(y)$$



The graph of $y = f^{-1}(x)$ is the reflection of the graph of f over the line $y = x$ (i.e. swap the axes). Further,

the domain of f is the range of f^{-1},

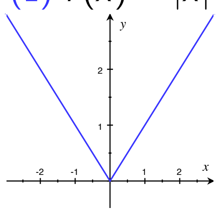
and

the range of f is the domain of f^{-1}.

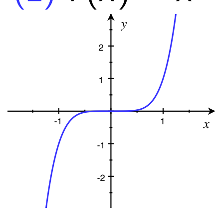
You try:

- For each of the following functions, (a) give the domain and range of f, and (b) decide if f is invertible.
- If f is invertible, then (c) sketch a graph of f^{-1}, (d) give the domain and range of f^{-1}, and (e) try to write a formula for f^{-1}.
- If f is not invertible over all of the real numbers, what is a restricted domain over which f is invertible? Over that restricted domain, do (c) and (d) from above.

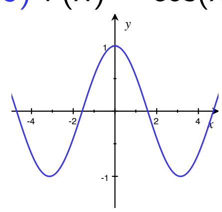
(1) $f(x) = |x|$



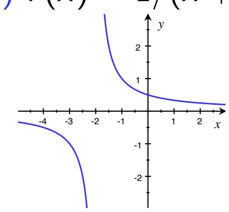
(2) $f(x) = x^5$



(3) $f(x) = \cos(x)$



(4) $f(x) = 1/(x+2)$



Calculating the inverse function algebraically

Given an invertible f, solve for f^{-1} by setting $f(y) = x$, and solving for $y = f^{-1}(x)$.

Example: Let $f(x) = 1/(x+2)$.

Set

$$x = f(y) = 1/(y+2).$$

Then

$$y + 2 = 1/x, \qquad \text{so that } \boxed{f^{-1}(x) = y = (1/x) - 2}.$$

Example: Let $f(x) = x^3 + 2$. (Check: is it invertible??)

Set

$$x = f(y) = y^3 + 2.$$

Then

$$y^3 = x - 2, \qquad \text{so that } \boxed{f^{-1}(x) = y = (x-2)^{1/3}}.$$

Checking your answer algebraically

Recall that f^{-1} is defined by

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Example: We calculated that if $f(x) = 1/(x+2)$, then $f^{-1}(x) = (1/x) - 2$. Let's check!

$$\begin{aligned} f(f^{-1}(x)) &= 1/((1/x) - 2 + 2) \\ &= 1/(1/x) = x \quad \checkmark \end{aligned}$$

and

$$\begin{aligned} f^{-1}(f(x)) &= (1/1/(x+2)) - 2 \\ &= x + 2 - 2 = x \quad \checkmark \end{aligned}$$

You try:

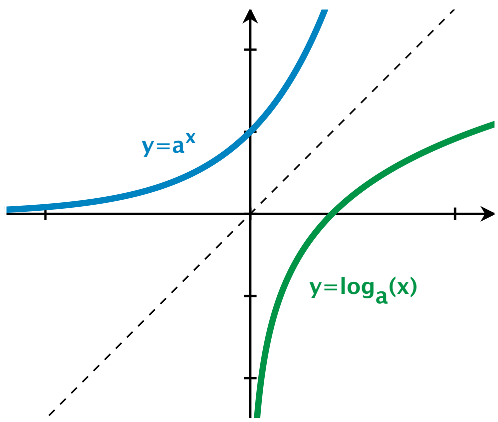
1. Check that if $f(x) = x^3 + 2$ then $f^{-1}(x) = (x-2)^{1/3}$ by calculating $f(f^{-1}(x))$ and $f^{-1}(f(x))$.
2. For the following functions, calculate $f^{-1}(x)$ and verify your answer as above. (a) $f(x) = 3/(x-1)$ (b) $f(x) = 5\sqrt{x-2}$

Logarithms

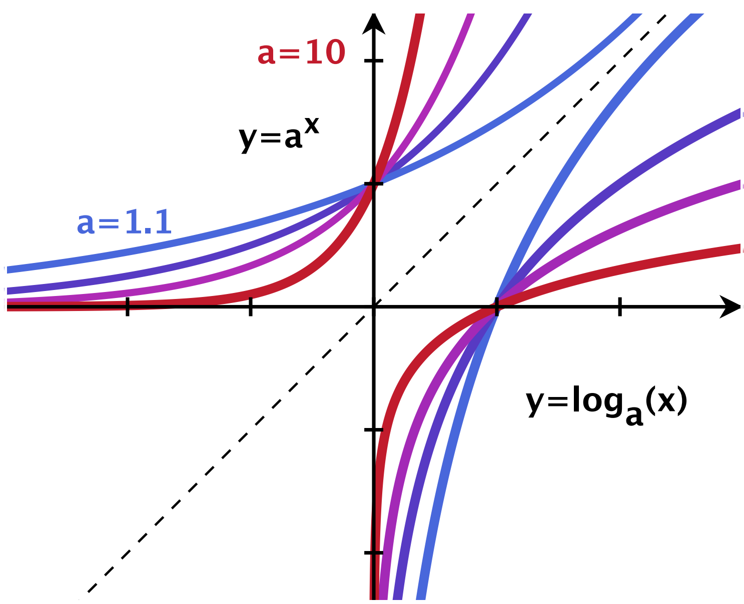
The exponential function a^x has inverse $\log_a(x)$, i.e.

$$\log_a(a^x) = x = a^{\log_a(x)}, \quad \text{i.e.}$$

$$y = a^x \quad \text{if and only if} \quad \log_a(y) = x.$$



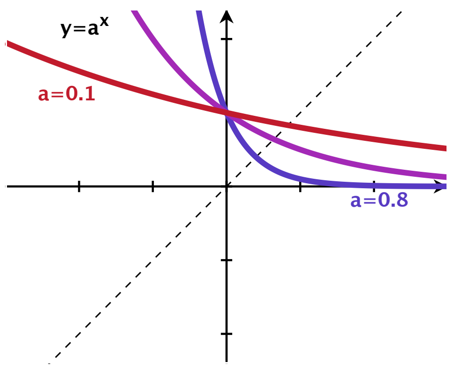
Properties of Logarithms

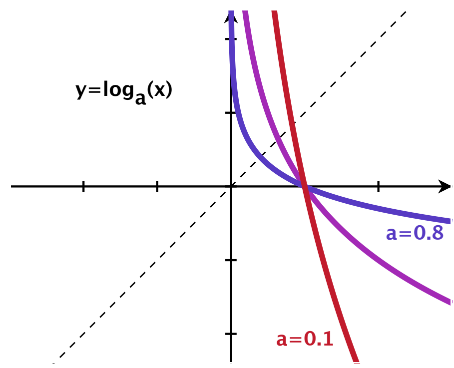


Domain: $(0, \infty)$ i.e. all $x > 0$ Range: $(-\infty, \infty)$ i.e. all x

Properties of Logarithms

$0 < a < 1$:





Domain: $(0, \infty)$ i.e. all $x > 0$ Range: $(-\infty, \infty)$ i.e. all x

Properties of Logarithms

Since...	we know...
1. $a^0 = 1$	1. $\log_a(1) = 0$
2. $a^1 = a$	2. $\log_a(a) = 1$
3. $a^b * a^c = a^{b+c}$	3. $\log_a(b * c) = \log_a(b) + \log_a(c)$
4. $(a^b)^c = a^{b*c}$	4. $\log_a(b^c) = c \log_a(b)$

Example: why $\log_a(b * c) = \log_a(b) + \log_a(c)$:
Suppose $y = \log_a(b) + \log_a(c)$.

Then $a^y = a^{\log_a(b)+\log_a(c)} = a^{\log_a(b)} a^{\log_a(c)} = b * c$.

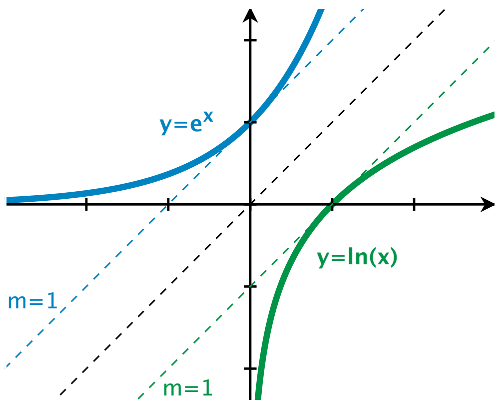
So $y = \log_a(b * c)$ as well!

Lastly: $\dfrac{\log_a(b)}{\log_a(c)} = \log_c(b)$

Favorite logarithmic function

Remember: $y = e^x$ is the function whose slope through the point (0,1) is 1.
The *inverse* to $y = e^x$ is the *natural log*:

$$\ln(x) = \log_e(x)$$



We will often use the facts that $e^{\ln(x)} = x$ (for $x > 0$) and $\ln(e^x) = x$ (for all x)

Two super useful facts:

Explain why:
(1) $\log_a(b) = \ln(b)/\ln(a)$

(2) $a^b = e^{b\ln(a)}$ [hint: start by rewriting $b\ln(a)$, and use the fact that $e^{\ln(x)} = x$]

Examples:

(1) Condense the logarithmic expressions

$$\frac{1}{2}\ln(x)+3\ln(x+1) \qquad 2\ln(x+5)-\ln(x) \qquad \frac{1}{3}(\log_3(x)-\log_3(x+1))$$

(2) Solve the following expressions for x:

$$e^{-x^2} = e^{-3x-4} \qquad 3(2^x) = 24$$

$$2(e^{3x-5}) - 5 = 11 \qquad \ln(3x+1) - \ln(5) = \ln(2x)$$

Inverse trig functions

Two notations:

$f(x)$	$f^{-1}(x)$
$\sin(x)$	$\sin^{-1}(x) = \arcsin(x)$
$\cos(x)$	$\cos^{-1}(x) = \arccos(x)$
$\tan(x)$	$\tan^{-1}(x) = \arctan(x)$
$\sec(x)$	$\sec^{-1}(x) = \text{arcsec}(x)$
$\csc(x)$	$\csc^{-1}(x) = \text{arccsc}(x)$
$\cot(x)$	$\cot^{-1}(x) = \text{arccot}(x)$

There are lots of points we know on these functions...

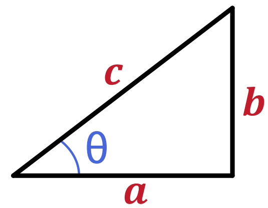
Examples:

1. Since $\sin(\pi/2) = 1$, we have $\arcsin(1) = \pi/2$
2. Since $\cos(\pi/2) = 0$, we have $\arccos(0) = \pi/2$

Etc...

In general:

arc___(-) takes in a ratio and spits out an angle:



$\cos(\theta) = a/c$ so $\arccos(a/c) = \theta$

$\sin(\theta) = b/c$ so $\arcsin(b/c) = \theta$

$\tan(\theta) = b/a$ so $\arctan(b/a) = \theta$

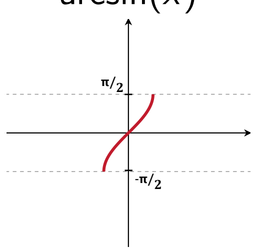
Domain problems:

$\sin(0) = 0, \quad \sin(\pi) = 0, \quad \sin(2\pi) = 0, \quad \sin(3\pi) = 0, \ldots$

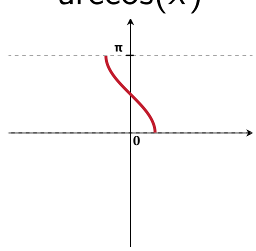
So which is the right answer to arcsin(0), really?

Graphs

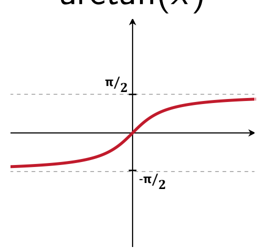
$\arcsin(x)$



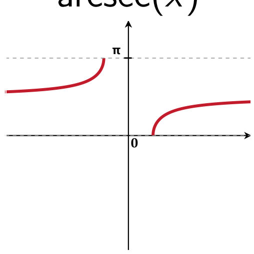
$\arccos(x)$



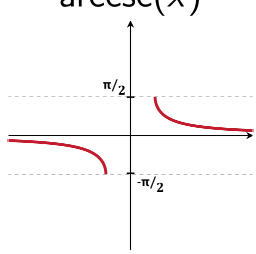
$\arctan(x)$



$\text{arcsec}(x)$



$\text{arccsc}(x)$



$\text{arccot}(x)$