Inverse trig functions

11/21/2011

Remember: $f^{-1}(x)$ is the inverse function of $f(x)$ if

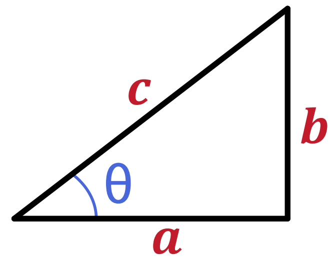
$$y = f(x) \qquad \text{implies} \qquad f^{-1}(y) = x.$$

For inverse functions to the trigonometric functions, there are two notations:

$f(x)$	$f^{-1}(x)$
$\sin(x)$	$\sin^{-1}(x) = \arcsin(x)$
$\cos(x)$	$\cos^{-1}(x) = \arccos(x)$
$\tan(x)$	$\tan^{-1}(x) = \arctan(x)$
$\sec(x)$	$\sec^{-1}(x) = \text{arcsec}(x)$
$\csc(x)$	$\csc^{-1}(x) = \text{arccsc}(x)$
$\cot(x)$	$\cot^{-1}(x) = \text{arccot}(x)$

In general:

arc___(-) takes in a ratio and spits out an angle:



$\cos(\theta) = a/c$ so $\arccos(a/c) = \theta$

$\sin(\theta) = b/c$ so $\arcsin(b/c) = \theta$

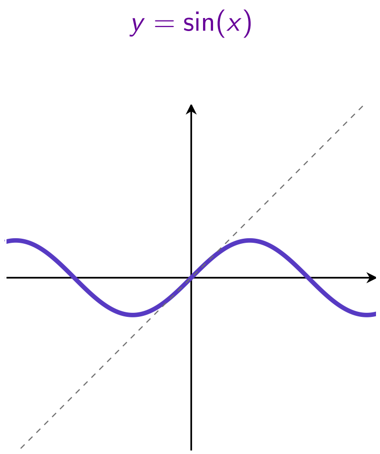
$\tan(\theta) = b/a$ so $\arctan(b/a) = \theta$

There are lots of points we know on these functions...

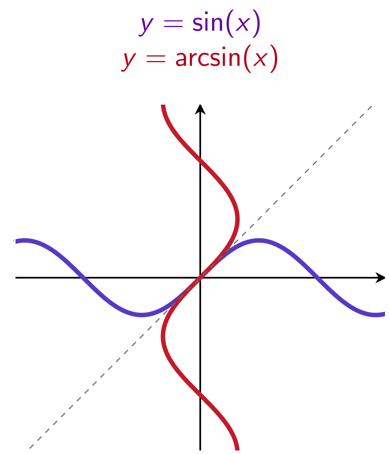
Examples:

1. Since $\sin(\pi/2) = 1$, we have $\arcsin(1) = \pi/2$
2. Since $\cos(\pi/2) = 0$, we have $\arccos(0) = \pi/2$
3. $\arccos(1) =$
4. $\arcsin(\sqrt{2}/2) =$
5. $\arctan(1) =$

Domain/range

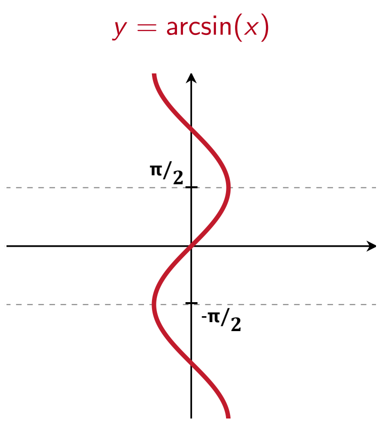


Domain/range



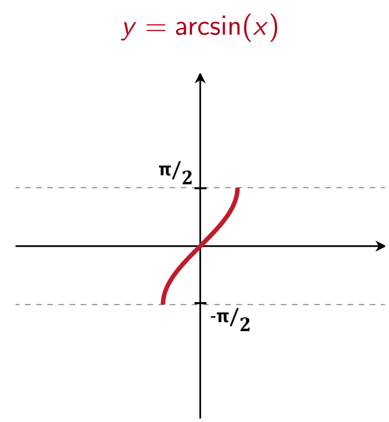
Domain: $-1 \leq x \leq 1$

Domain/range



Domain: $-1 \leq x \leq 1$

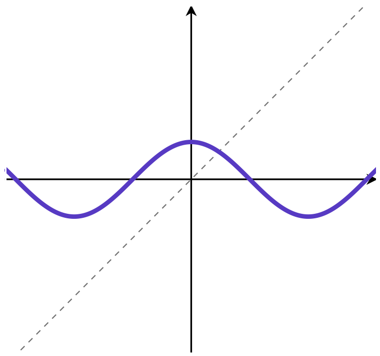
Domain/range



Domain: $-1 \leq x \leq 1$ Range: $-\pi/2 \leq y \leq \pi/2$

Domain/range

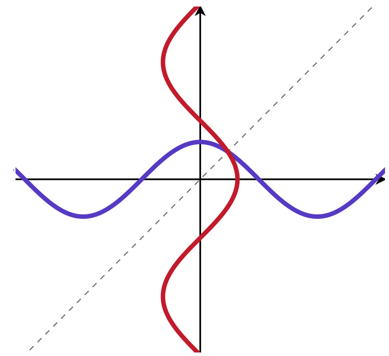
$y = \cos(x)$



Domain/range

$y = \cos(x)$

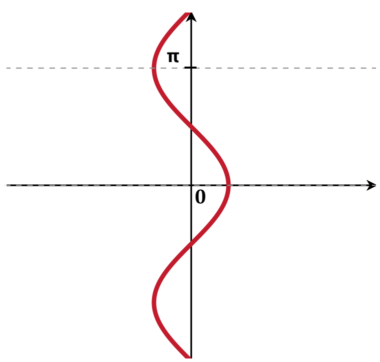
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

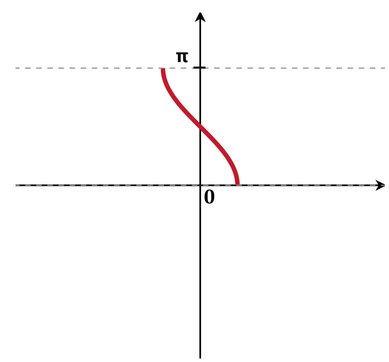
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

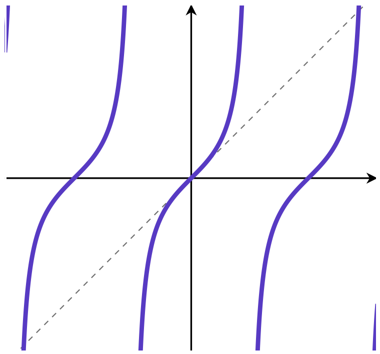
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$ Range: $0 \leq y \leq \pi$

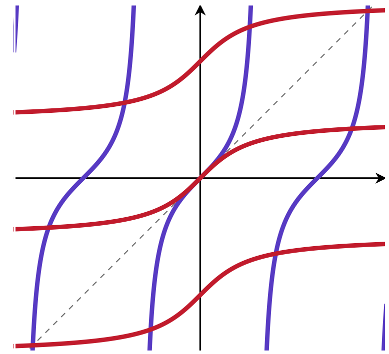
Domain/range

$y = \tan(x)$



Domain/range

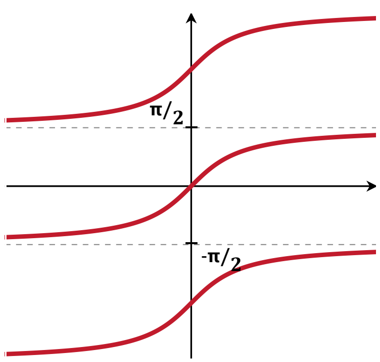
$y = \tan(x)$
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

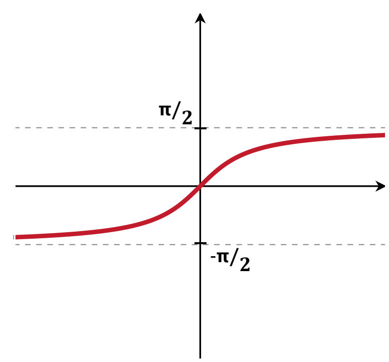
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

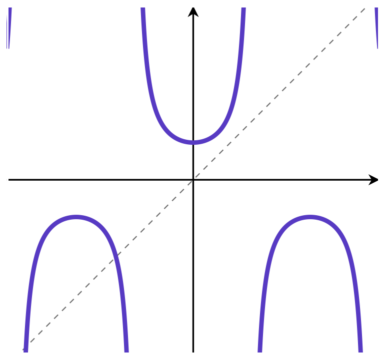
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$ Range: $-\pi/2 < y < \pi/2$

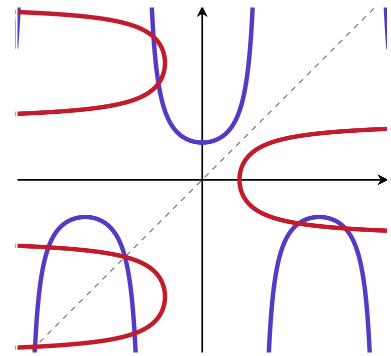
Domain/range

$y = \sec(x)$



Domain/range

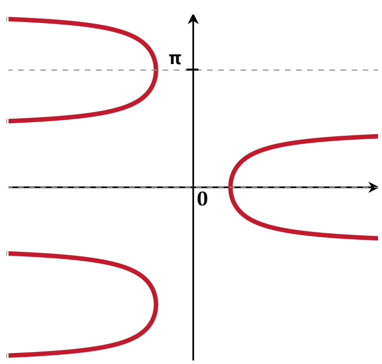
$y = \sec(x)$
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

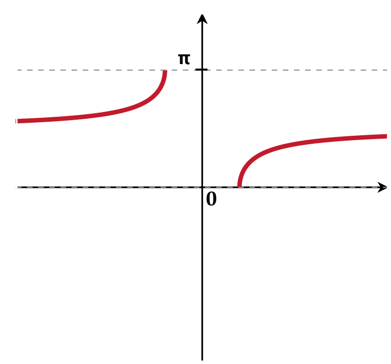
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

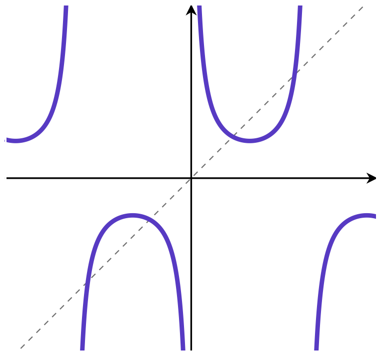
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$ Range: $0 \leq y \leq \pi$

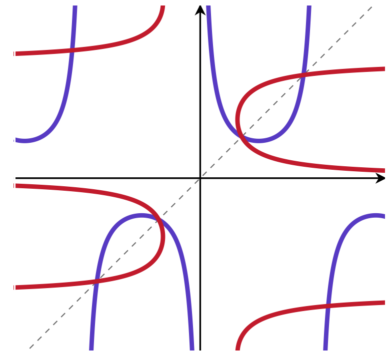
Domain/range

$y = \csc(x)$



Domain/range

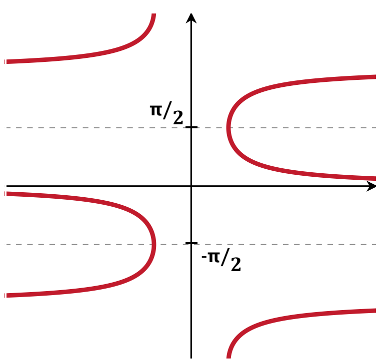
$y = \csc(x)$
$y = \operatorname{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

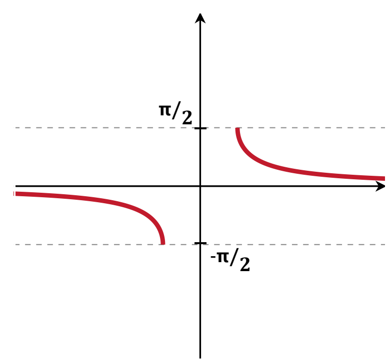
$y = \operatorname{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

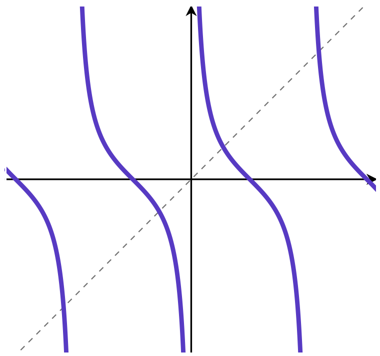
$y = \operatorname{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$ Range: $-\pi/2 \leq y \leq \pi/2$

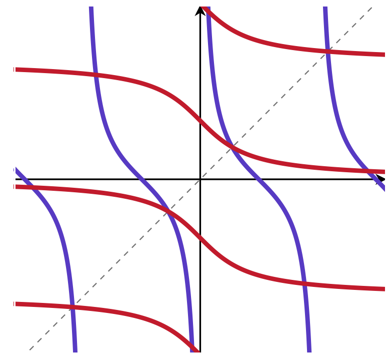
Domain/range

$y = \cot(x)$



Domain/range

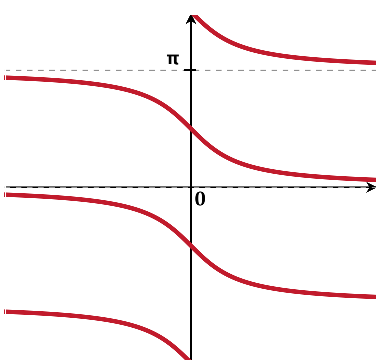
$y = \cot(x)$
$y = \operatorname{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

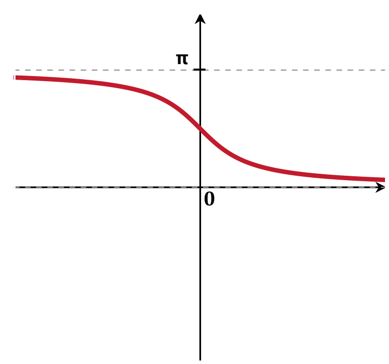
$y = \operatorname{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

$y = \operatorname{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$ Range: $0 < y < \pi$

Derivatives

Use implicit differentiation (just like $\ln(x)$).

Q. Let $y = \arcsin(x)$. What is $\frac{dy}{dx}$?

If $y = \arcsin(x)$ then $x = \sin(y)$.

Take $\frac{d}{dx}$ of both sides of $x = \sin(y)$:

LHS: $\frac{d}{dx} x = 1$ RHS: $\frac{d}{dx} \sin(y) = \cos(y) \frac{dy}{dx} = \cos(\arcsin(x)) \frac{dy}{dx}$

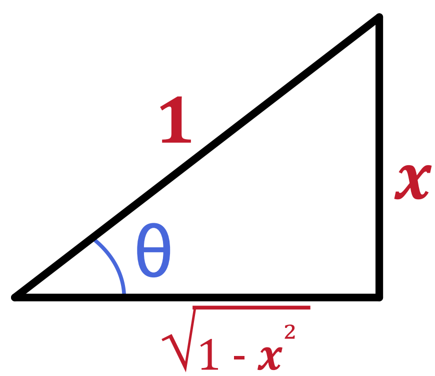
So

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))}.}$$

Simplifying $\cos(\arcsin(x))$

Call $\arcsin(x) = \theta$.

$\sin(\theta) = x$

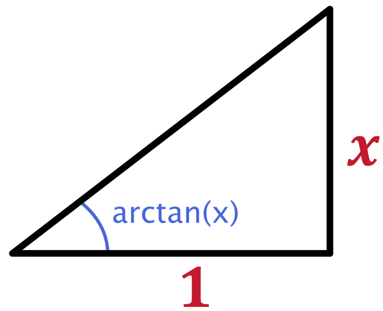


So $\cos(\arcsin(x)) = \sqrt{1 - x^2}$

So $$\frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - x^2}}.$$

Calculate $\frac{d}{dx}\arctan(x)$.

1. Rewrite $y = \arctan(x)$ as $x = \tan(y)$.
2. Use implicit differentiation and solve for $\frac{dy}{dx}$.
3. Your answer will have $\sec(\arctan(x))$ in it. Simplify this expression using



Recall: In general, if $y = f^{-1}(x)$, then $x = f(y)$.

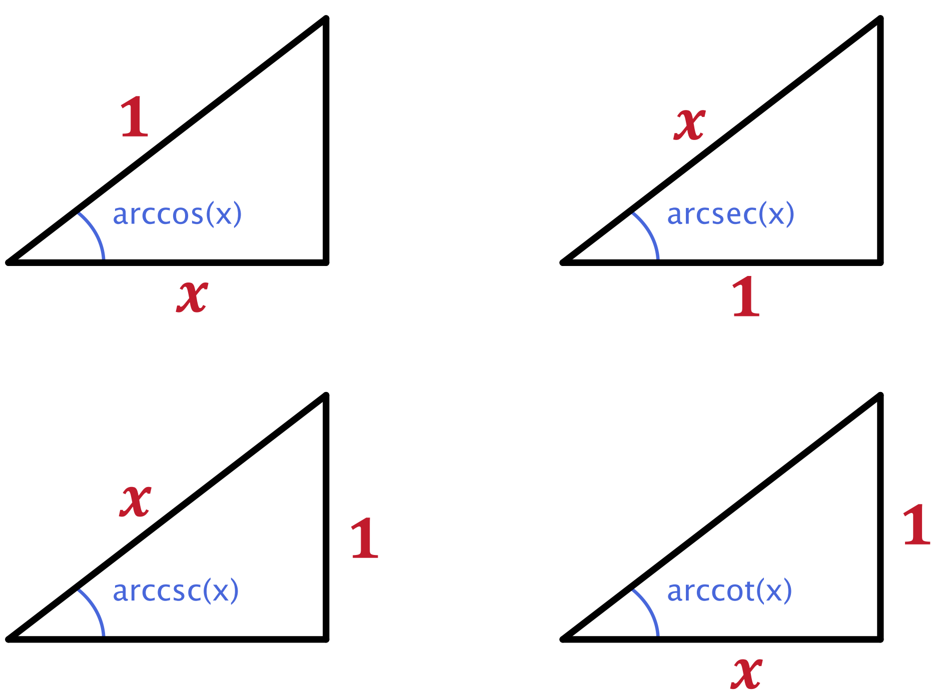
$$\text{So } 1 = f'(y)\frac{dy}{dx} = f'\left(f^{-1}(x)\right), \text{ and so}$$

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f(x))}}$$

$f(x)$	$f'(x)$
$\cos(x)$	$-\sin(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$

$f(x)$	$f'(x)$
$\arctan(x)$	$-\frac{1}{\sin(\arccos(x))}$
$\text{arcsec}(x)$	$\frac{1}{\sec(\text{arcsec}(x))\tan(\text{arcsec}(x))}$
$\text{arccsc}(x)$	$-\frac{1}{\csc(\text{arccsc}(x))\cot(\text{arccsc}(x))}$
$\text{arccot}(x)$	$-\frac{1}{\left(\csc(\text{arccot}(x))\right)^2}$

To simplify, use the triangles



More examples

Since $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, we know

1. $\frac{d}{dx}\arctan(\ln(x)) =$
2. $\displaystyle\int \frac{1}{1+x^2}dx =$
3. $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}dx =$