# Inverse trig functions

11/21/2011

Remember: $f^{-1}(x)$ is the inverse function of $f(x)$ if

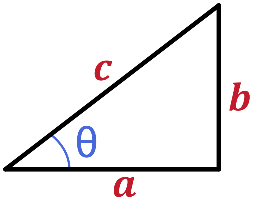
$$y = f(x) \qquad \text{implies} \qquad f^{-1}(y) = x.$$

For inverse functions to the trigonometric functions, there are two notations:

| $f(x)$ | $f^{-1}(x)$ |
|---|---|
| $\sin(x)$ | $\sin^{-1}(x) = \arcsin(x)$ |
| $\cos(x)$ | $\cos^{-1}(x) = \arccos(x)$ |
| $\tan(x)$ | $\tan^{-1}(x) = \arctan(x)$ |
| $\sec(x)$ | $\sec^{-1}(x) = \text{arcsec}(x)$ |
| $\csc(x)$ | $\csc^{-1}(x) = \text{arccsc}(x)$ |
| $\cot(x)$ | $\cot^{-1}(x) = \text{arccot}(x)$ |

In general:

arc___( - ) takes in a ratio and spits out an angle:



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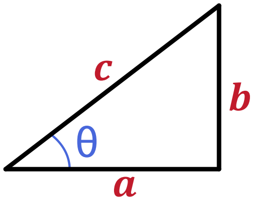


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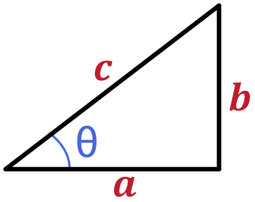


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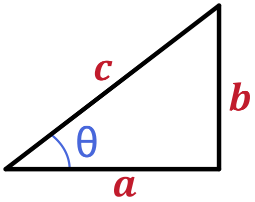



$\cos(\theta) = a/c$ so $\arccos(a/c) = \theta$

$\sin(\theta) = b/c$ so $\arcsin(b/c) = \theta$

$\tan(\theta) = b/a$ so $\arctan(b/a) = \theta$

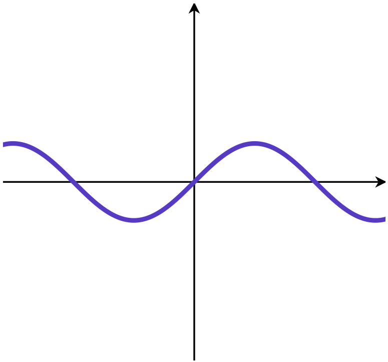
There are lots of points we know on these functions...

Examples:

1. Since $\sin(\pi/2) = 1$, we have $\arcsin(1) = \pi/2$
2. Since $\cos(\pi/2) = 0$, we have $\arccos(0) = \pi/2$
3. $\arccos(1) =$
4. $\arcsin(\sqrt{2}/2) =$
5. $\arctan(1) =$

# Domain/range

$$y = \sin(x)$$



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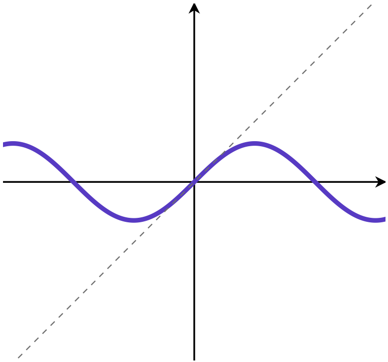

# Domain/range

$y = \sin(x)$
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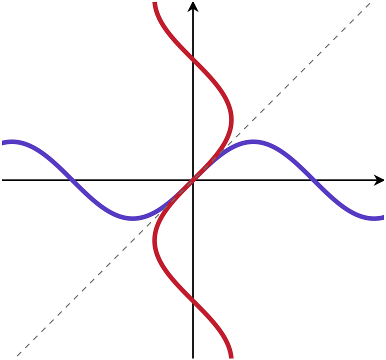

Domain: $-1 \leq x \leq 1$

# Domain/range

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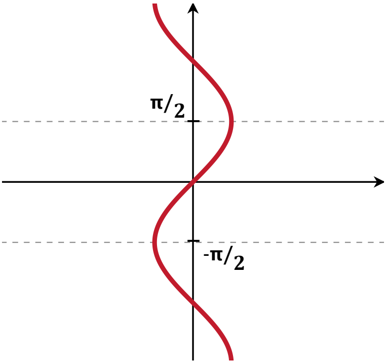


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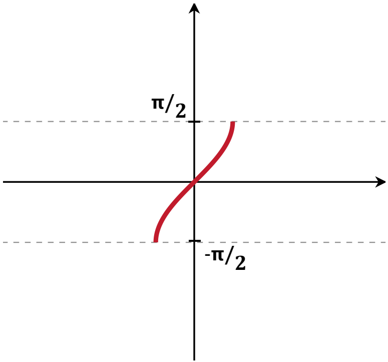
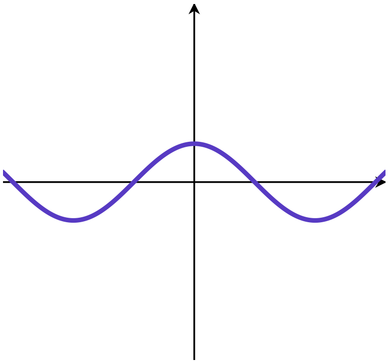



Domain: $-1 \leq x \leq 1$ Range: $-\pi/2 \leq y \leq \pi/2$

# Domain/range

$y = \cos(x)$



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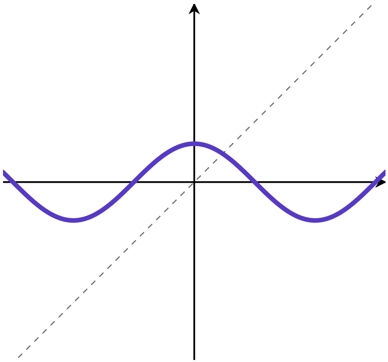

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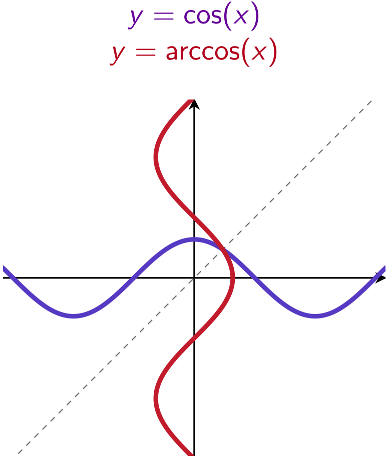


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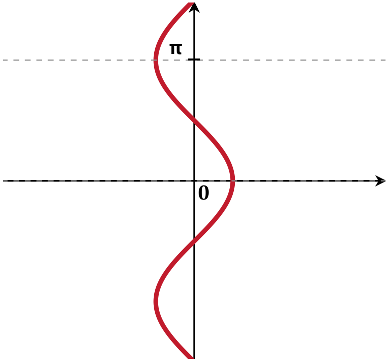
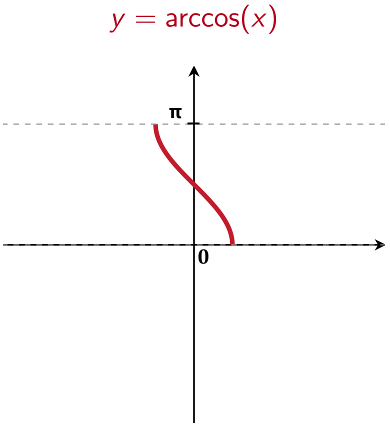
# Domain/range

$y = \arccos(x)$




Domain: $-1 \leq x \leq 1$

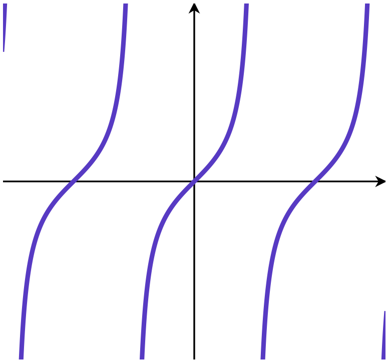
# Domain/range




Domain: $-1 \leq x \leq 1$ Range: $0 \leq y \leq \pi$

# Domain/range

$$y = \tan(x)$$



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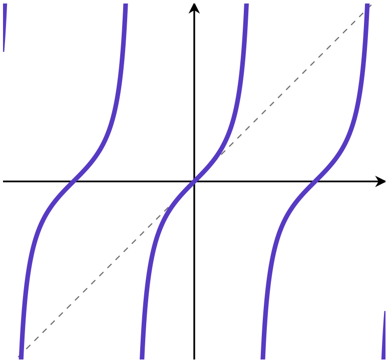
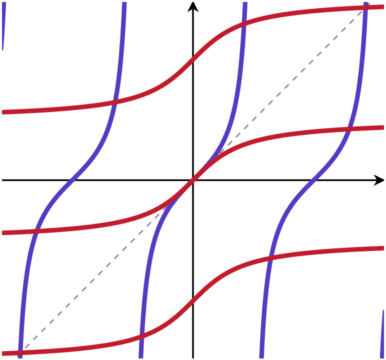


# Domain/range

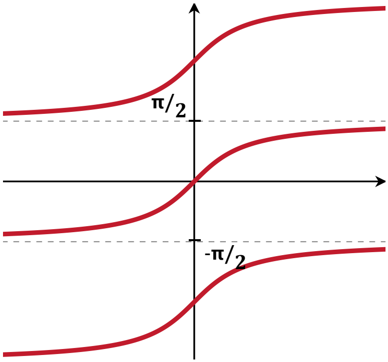
$y = \tan(x)$
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

# Domain/range

$y = \arctan(x)$

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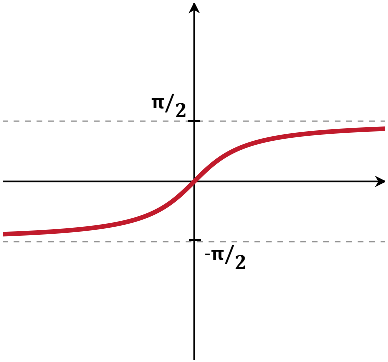
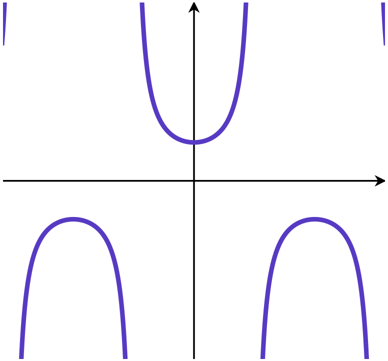



Domain: $-\infty \leq x \leq \infty$ Range: $-\pi/2 < y < \pi/2$

# Domain/range

$y = \sec(x)$



# Domain/range

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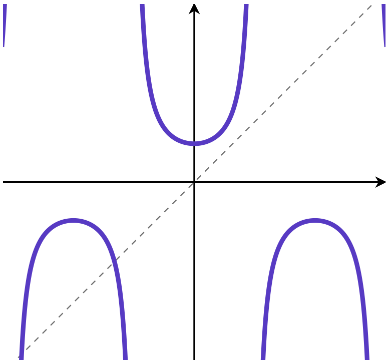
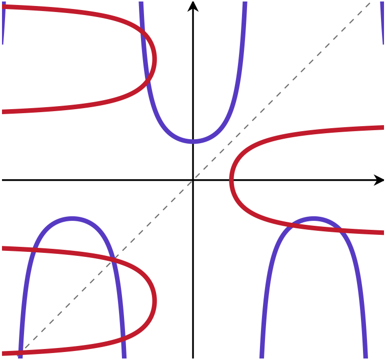


# Domain/range

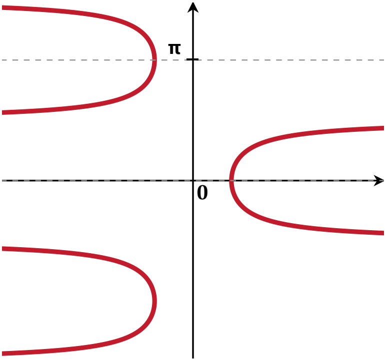
$y = \sec(x)$
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

# Domain/range

$y = \text{arcsec}(x)$

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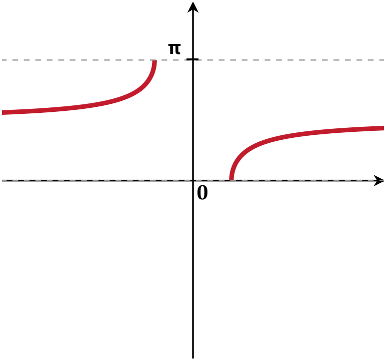
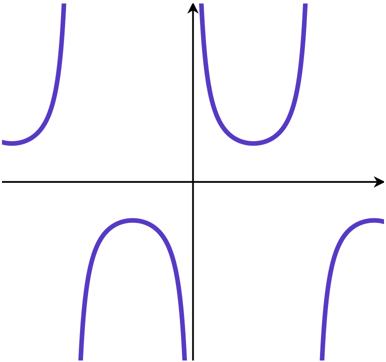



Domain: $x \le -1$ and $1 \le x$ Range: $0 \le y \le \pi$

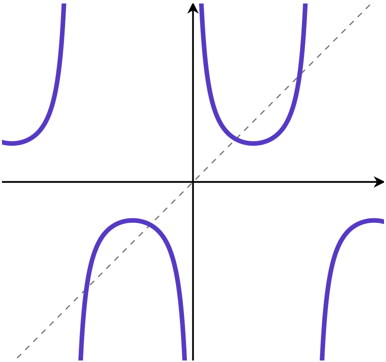
# Domain/range

$y = \csc(x)$



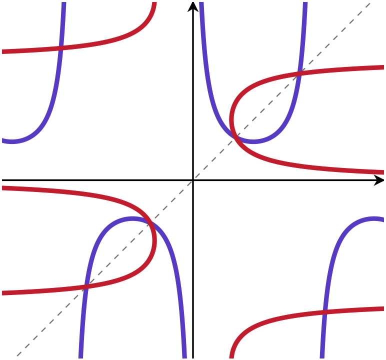
# Domain/range

$$y = \csc(x)$$



# Domain/range

$y = \csc(x)$
$y = \text{arccsc}(x)$



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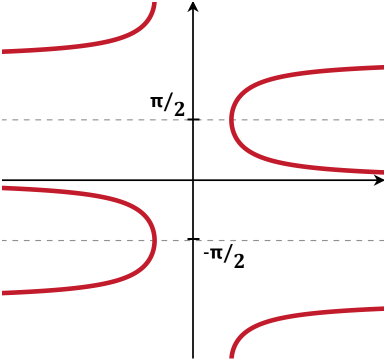


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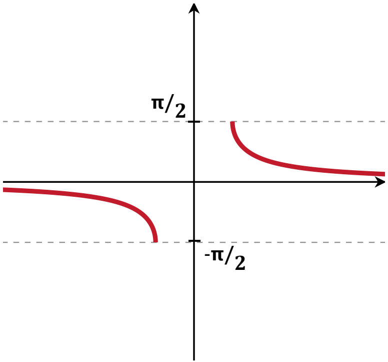
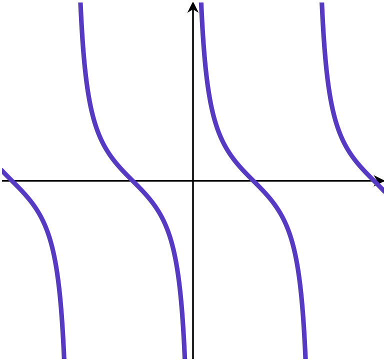



Domain: $x \leq -1$ and $1 \leq x$ Range: $-\pi/2 \leq y \leq \pi/2$

# Domain/range

$$y = \cot(x)$$



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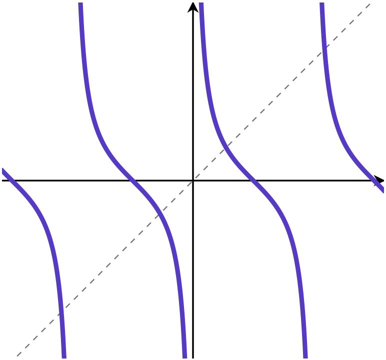

# Domain/range

$y = \cot(x)$

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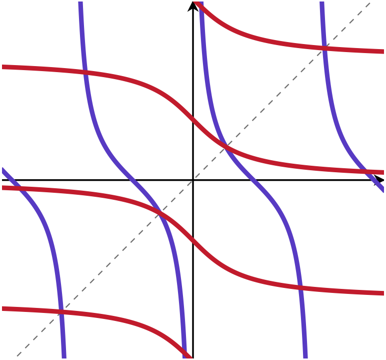

Domain: $-\infty \leq x \leq \infty$

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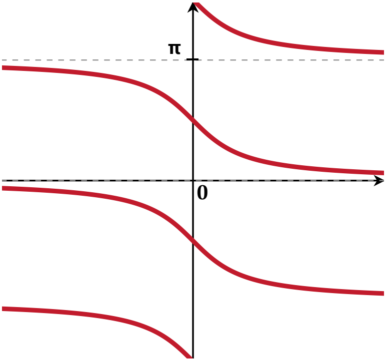


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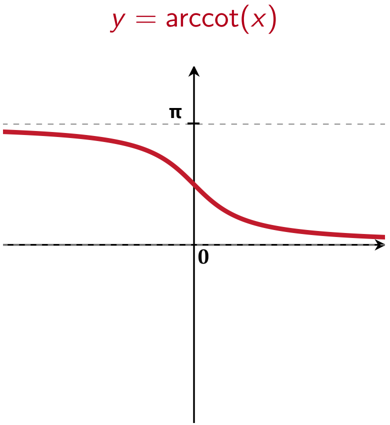
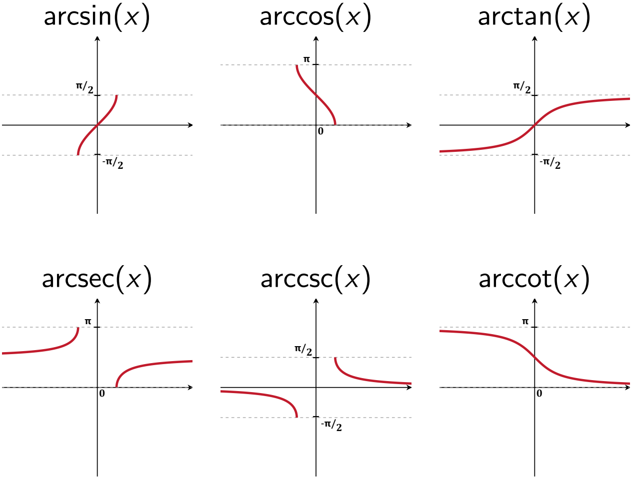



Domain: $-\infty \leq x \leq \infty$ Range: $0 < y < \pi$

# Graphs



arcsin(x)
π/2
-π/2
arccos(x)
π
0
arctan(x)
π/2
-π/2
arcsec(x)
π
0
arccsc(x)
π/2
-π/2
arccot(x)
π
0

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Use implicit differentiation (just like $\ln(x)$).

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LHS: $\dfrac{d}{dx}x = 1$ RHS: $\dfrac{d}{dx}\sin(y) = \cos(y)\dfrac{dy}{dx} = \cos(\arcsin(x))\dfrac{dy}{dx}$

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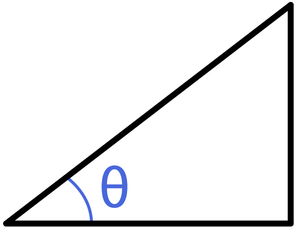
So

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))}.}$$

# Simplifying cos(arcsin($x$))

Call arcsin($x$) $= \theta$.



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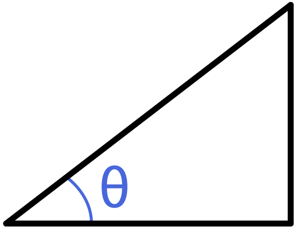
$\sin(\theta) = x$



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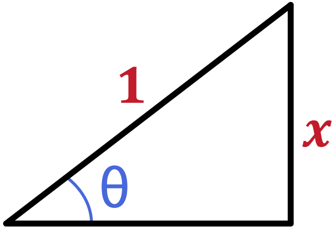

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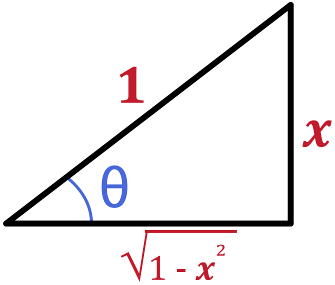

# Simplifying $\cos(\arcsin(x))$

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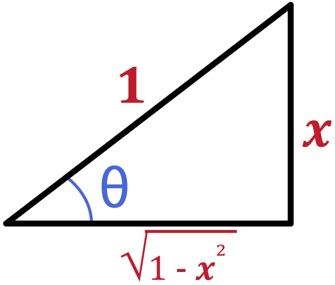


So $\cos(\theta) = \sqrt{1 - x^2}/1$

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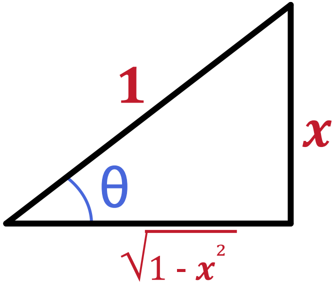


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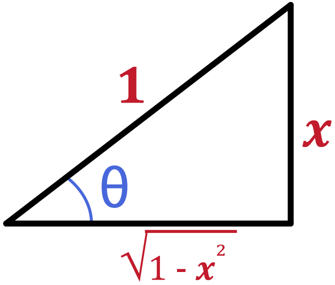
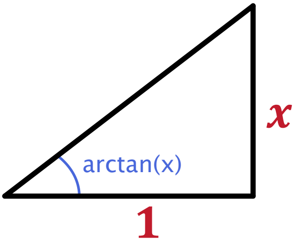



So $\cos(\arcsin(x)) = \sqrt{1 - x^2}$

So $\frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - x^2}}.$

# Calculate $\frac{d}{dx} \arctan(x)$.

1. Rewrite $y = \arctan(x)$ as $x = \tan(y)$.
2. Use implicit differentiation and solve for $\frac{dy}{dx}$.
3. Your answer will have $\sec(\arctan(x))$ in it. Simplify this expression using



Recall: In general, if $y = f^{-1}(x)$, then $x = f(y)$.

So $1 = f'(y)\frac{dy}{dx} = f'\left(f^{-1}(x)\right)$, and so

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f(x))}}$$

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| $f(x)$ | $f'(x)$ |
|---|---|
| $\cos(x)$ | $-\sin(x)$ |
| $\sec(x)$ | $\sec(x)\tan(x)$ |
| $\csc(x)$ | $-\csc(x)\cot(x)$ |
| $\cot(x)$ | $-\csc^2(x)$ |

Recall: In general, if $y = f^{-1}(x)$, then $x = f(y)$.

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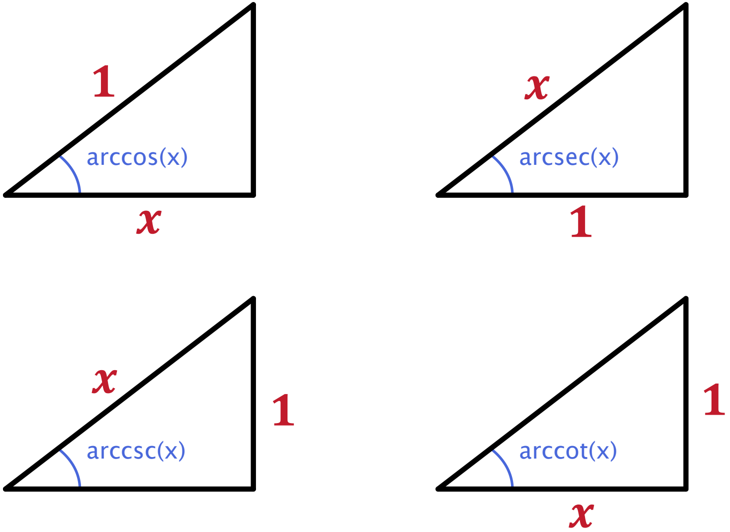
| $\cot(x)$ | $-\csc^2(x)$ |

| $f(x)$ | $f'(x)$ |
|---|---|
| $\arctan(x)$ | $-\frac{1}{\sin(\arccos(x))}$ |
| $\operatorname{arcsec}(x)$ | $\frac{1}{\sec(\operatorname{arcsec}(x))\tan(\operatorname{arcsec}(x))}$ |
| $\operatorname{arccsc}(x)$ | $-\frac{1}{\csc(\operatorname{arccsc}(x))\cot(\operatorname{arccsc}(x))}$ |
| $\operatorname{arccot}(x)$ | $-\frac{1}{\left(\csc(\operatorname{arccot}(x))\right)^2}$ |

To simplify, use the triangles



# More examples

Since $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, we know

1. $\frac{d}{dx}\arctan(\ln(x)) =$

2. $\displaystyle\int \frac{1}{1+x^2}dx =$

3. $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}dx =$