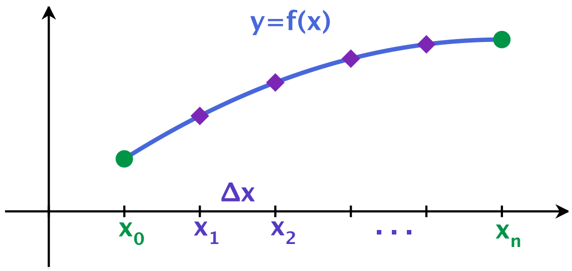
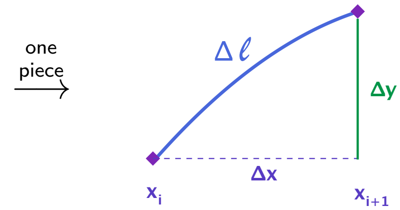
Arc length 11/18/2011

Suppose you want to know what the length of a curve $y = f(x)$ is from the point $(a, f(a))$ to the point $(b, f(b))$:



Slice!

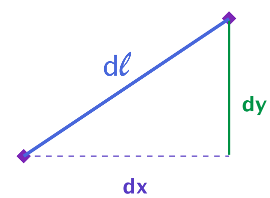


Let n go to ∞

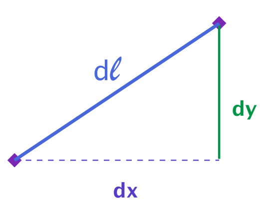
$\downdownarrows$

$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta \ell)_i = \int_{x=a}^{x=b} d\ell$$

$$\boxed{d\ell = \sqrt{dx^2 + dy^2}}$$



Manipulating into something we can actually calculate...



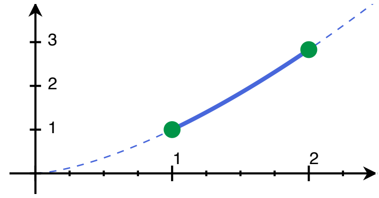
Remember, $y = f(x)$.

$$\begin{aligned} d\ell = \sqrt{dx^2 + dy^2} &= \sqrt{dx^2 + dy^2}\,\frac{dx}{dx} \\ &= \sqrt{\frac{dx^2 + dy^2}{dx^2}}\,dx = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}}\,dx \\ &= \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1 + (f'(x))^2}\,dx \end{aligned}$$

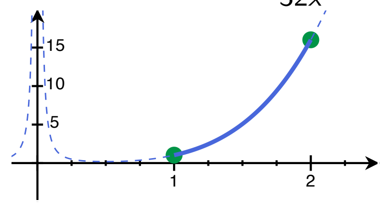
So $$\boxed{\ell = \int_{x=a}^{b} \sqrt{1 + (f'(x))^2}\,dx}$$

Example

Find the length of the arc $y = x^{3/2}$, from $x = 0$ to $x = 1$.



Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from x = 1 to x = 2.



$$f(x) = x^4 + \frac{1}{32}x^{-2} \implies f'(x) = 4x^3 - \frac{1}{16}x^{-3} = \frac{64x^6 - 1}{16x^3}$$

Keeping the algebra tame:
Let $A = (2x)^3 = 8x^3$ and so $A^2 = 64x^6$, and $f'(x) = \frac{A^2-1}{2A}$.
So

$$1+(f'(x))^2 = 1+\left(\frac{A^2-1}{2A}\right)^2$$

Most of the time, the resulting integral is "hard" (not elementary)

Set up (but do not integrate) the integrals which compute the length of the following functions:

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
5. $f(x) = e^x$ from $x = 0$ to 1
6. $f(x) = \sqrt{1 - x^2}$ from $x = -1$ to 1