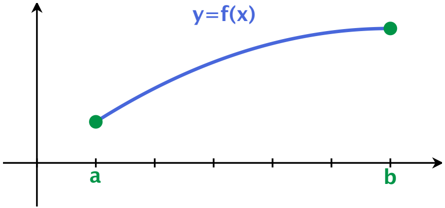
Arc Length

11/18/2011

Suppose you want to know what the length of a curve $y = f(x)$ is from the point $(a, f(a))$ to the point $(b, f(b))$:



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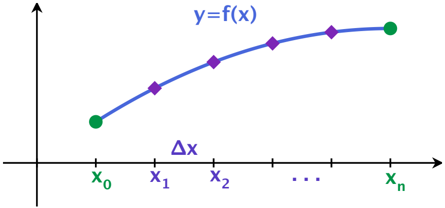

Slice!

$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} (\text{little length})_i$$

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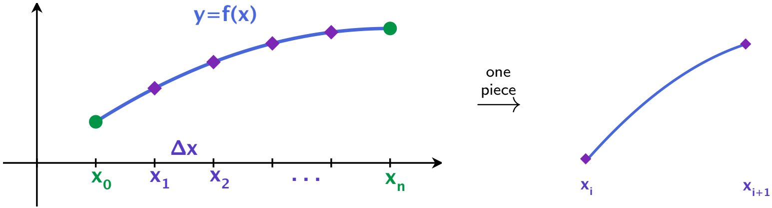

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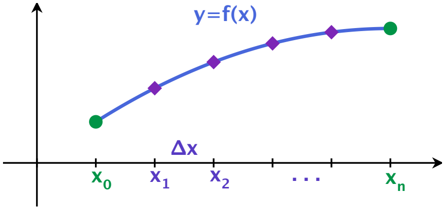
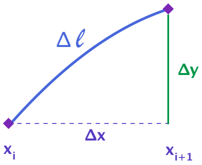


one piece $\longrightarrow$



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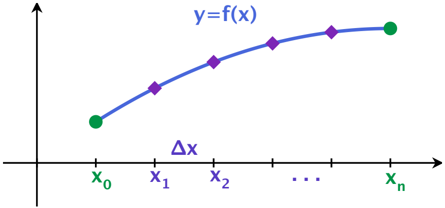

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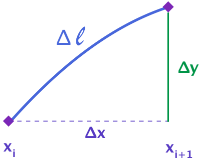

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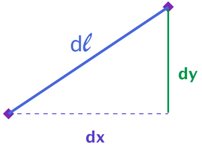
Let n go to ∞

$\downdownarrows$



$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta\ell)_i = \int_{x=a}^{x=b} d\ell$$

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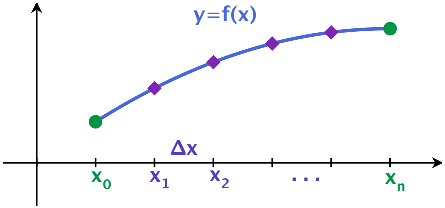

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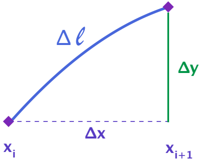

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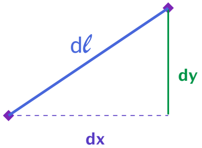
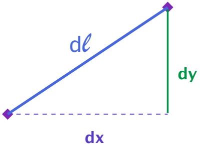


$$\ell = \lim_{n \to \infty} \sum_{i=1}^{n} (\Delta \ell)_i = \int_{x=a}^{x=b} d\ell$$

$$\boxed{d\ell = \sqrt{dx^2 + dy^2}}$$

Manipulating into something we can actually calculate...



Remember, $y = f(x)$.

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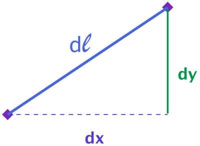

Remember, $y = f(x)$.

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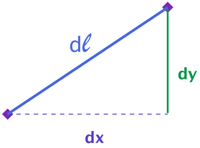

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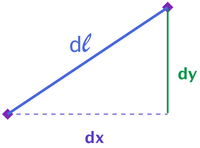

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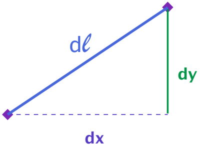

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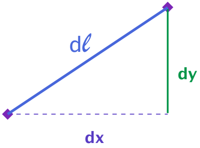

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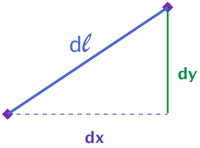

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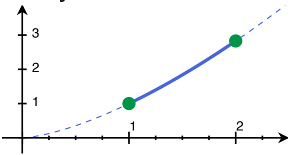
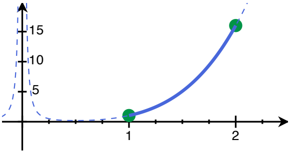
So $$\boxed{\ell = \int_{x=a}^{b} \sqrt{1 + \left(f'(x)\right)^2}\ dx}$$

Example

Find the length of the arc $y = x^{3/2}$, from $x = 0$ to $x = 1$.



Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from x = 1 to x = 2.



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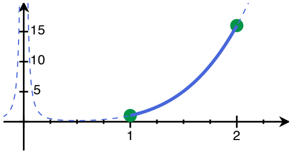

$$f(x) = x^4 + \frac{1}{32}x^{-2} \implies f'(x) = 4x^3 - \frac{1}{16}x^{-3} = \frac{64x^6 - 1}{16x^3}$$

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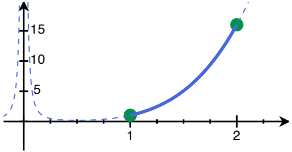

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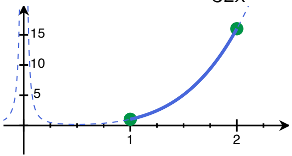
Keeping the algebra tame:
Let $A = (2x)^3 = 8x^3$ and so $A^2 = 64x^6$, and $f'(x) = \frac{A^2-1}{2A}$.

Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from x = 1 to x = 2.



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So

$$1+(f'(x))^2 = 1+\left(\frac{A^2-1}{2A}\right)^2$$

Most of the time, the resulting integral is "hard" (not elementary)

Set up (but do not integrate) the integrals which compute the length of the following functions:

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
5. $f(x) = e^x$ from $x = 0$ to 1
6. $f(x) = \sqrt{1 - x^2}$ from $x = -1$ to 1