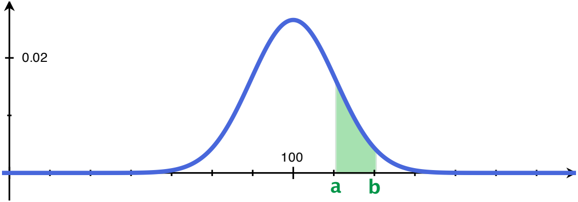
Numerical integration example

The probability that someone's IQ falls between a and b is given by the area under the curve

$$D(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2*(15)^2}}$$

(this is the normal distribution with average 100 and standard deviation 15)



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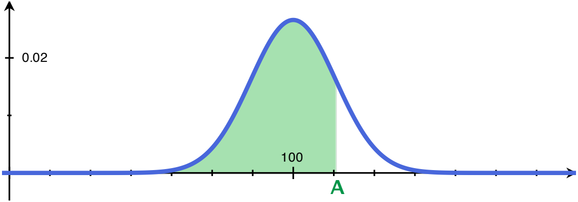

If someone has an IQ of A, they're approximately in the percentile:

$$\int_0^A D(x)dx$$

$$\left(\int_{-\infty}^0 D(x)dx \approx 0\right)$$

Q. If you have an IQ of 130, what percentile are you in?

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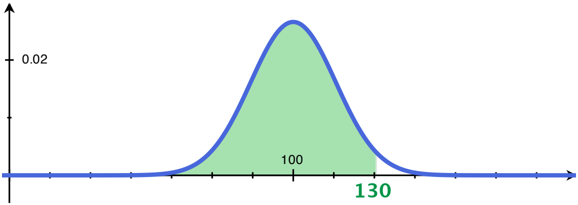
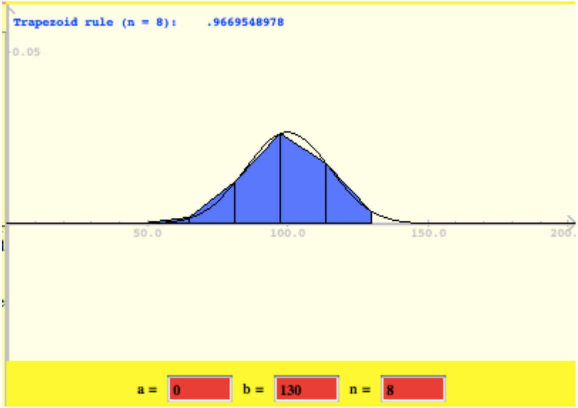
A. $\displaystyle\int_0^{130} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2*(15)^2}}\, dx$



Q. If you have an IQ of 130, what percentile are you in?

A. $$\int_0^{130} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2*(15)^2}}\, dx \approx 96.665\%$$

Trapezoids with $n = 8$:



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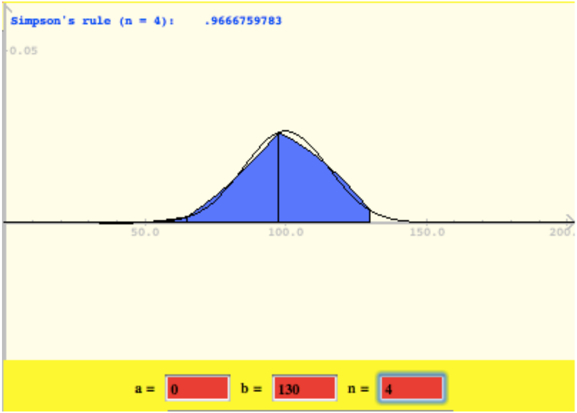
Simpsons with $n = 8$:



How good are these estimates?

The error for each of these estimates can be bounded!
Suppose you have approximated $\int_a^b f(x)\, dx$....

For **Trapezoids**, the error is bounded in terms of the second derivative of the function.

For **Simpson's rule**, the error is bounded in terms of the fourth derivative of the function.

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For **Trapezoids**, the error is bounded in terms of the second derivative of the function.

$$\boxed{\text{error}(n \text{ Trapezoids}) \leq M_2(b-a)^3/12 * n^2}$$

where M_2 is the maximum value of $f''(x)$ over the interval $[a, b]$

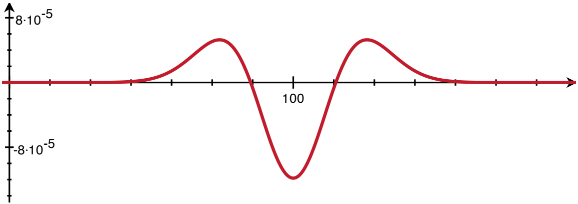
For **Simpson's rule**, the error is bounded in terms of the fourth derivative of the function.

$$\boxed{\text{error}(n \text{ subintervals, i.e. } \tfrac{n}{2} \text{ parabolas}) \leq M_4(b-a)^5/180 * n^4}$$

where M_4 is the maximum value of $f^{(4)}(x)$ over the interval $[a, b]$

$$f(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2*(15)^2}}$$

$$f''(x) = Ke^{-1/450(x-100)^2}(x^2 - 200x + 9775),$$ where $K = \frac{1}{759375\sqrt{2\pi}}$



$$M_2 = 0.00005275$$

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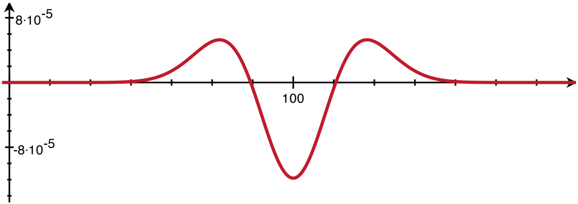


$$M_2 = 0.00005275$$

So, the error in the Trapezoid approximation of

$$\int_0^{130} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2*(15)^2}} \, dx$$

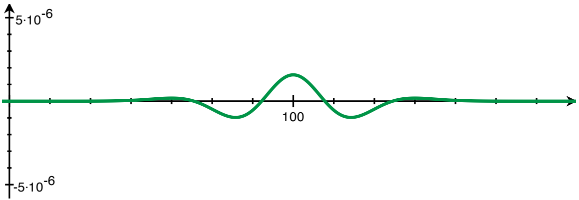
with 8 trapezoids can't be any larger than

$$0.00005275 * 130^3/(12 * 8^2) \approx \boxed{0.1509}$$

$$f(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2*(15)^2}}$$

$$f^{(4)}(x) = Ke^{-1/450(x-100)^2}(86651875 - 3730000x + 58650x^2 - 400x^3 + x^4),$$

where $K = \frac{1}{38443359375\sqrt{2\pi}}$



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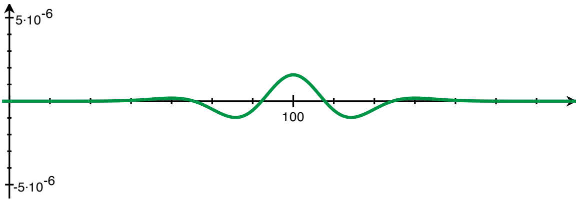


$$M_4 = 0.000001576$$

So, the error in the Simpson's approximation of

$$\int_0^{130} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2*(15)^2}} \, dx$$

with 8 subintervals, i.e. 4 parabolas, can't be any larger than

$$0.000001576 * 130^5/(180 * 8^4) \approx \boxed{0.07937}$$

Example

Suppose we approximated $\int_0^5 \ln(x+1)\,dx$ using

(a) Trapezoids with $n = 3$, and

(b) Simpson's rule with $n = 2$.

Which is guaranteed to be the better approximation?

Example

Suppose we approximated $\int_0^5 \ln(x+1)\, dx$ using

(a) Trapezoids with $n = 3$, and

(b) Simpson's rule with $n = 2$.

Which is guaranteed to be the better approximation?

Strategy:

1. Calculate the second and fourth derivatives of $f(x)$.
2. Maximize $f''(x)$ over the interval $[0, 5]$. Call its maximum value M_2.
3. Plug M_2, $(b - a)$ and n into the error bound formula for Trapezoids.
4. Maximize $f^{(4)}(x)$ over the interval $[0, 5]$.
   Call its maximum value M_4.
5. Plug M_4, $(b - a)$ and n into the error bound formula Simpson's rule.
6. Compare.

Area between curves

11/16/2011

Areas Between Curves

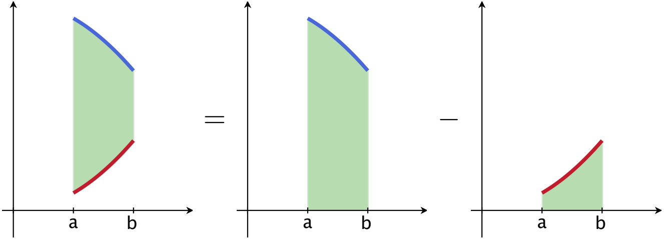
We know that if f is a continuous nonnegative function on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the area under the graph of f and above the interval.

Now suppose we are given two continuous functions, $f(x)$ and $g(x)$ so that $g(x) \leq f(x)$ for all x in the interval $[a, b]$.

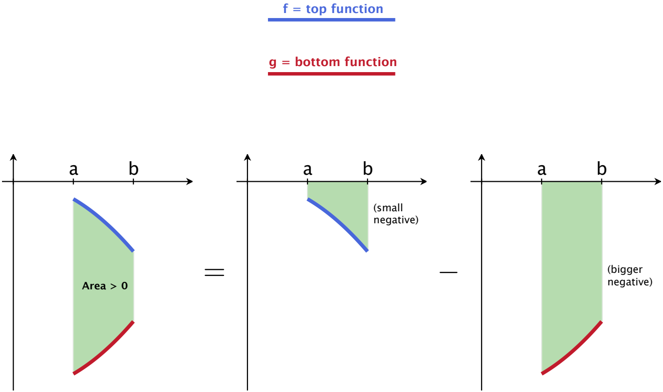
How do we find the area bounded by the two functions over that interval?

f = top function

g = bottom function



$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$



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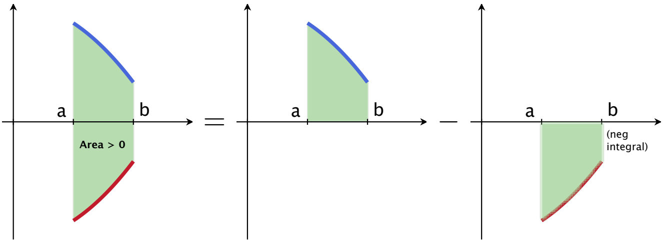

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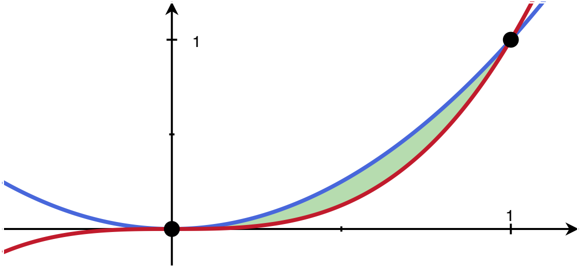
Example

Find the area of the region between the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$.



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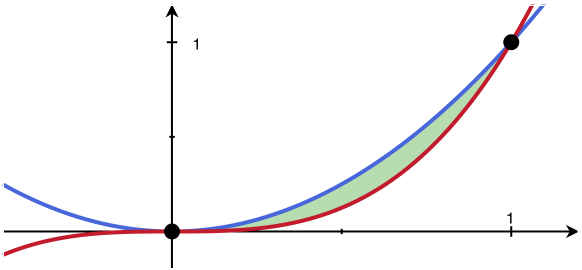

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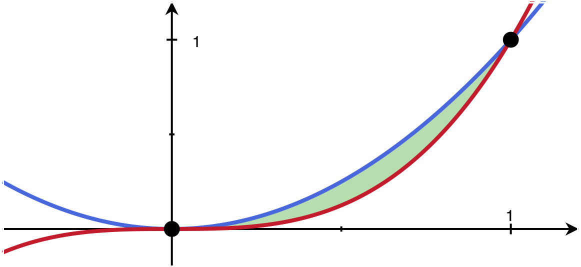

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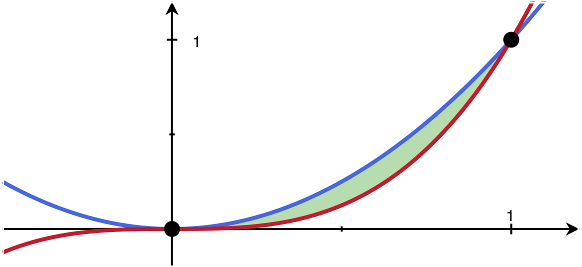

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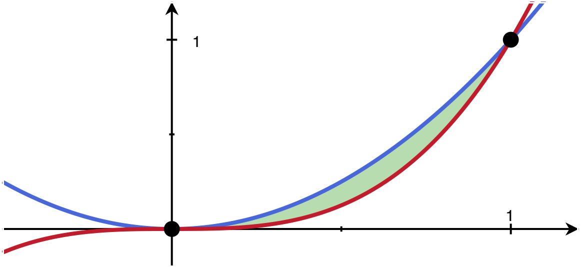

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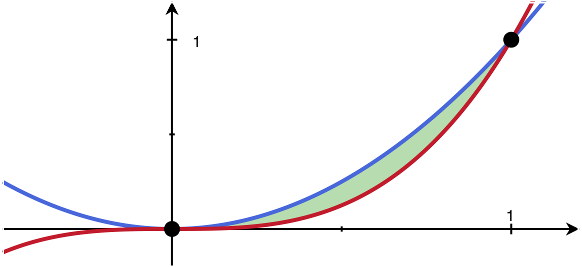

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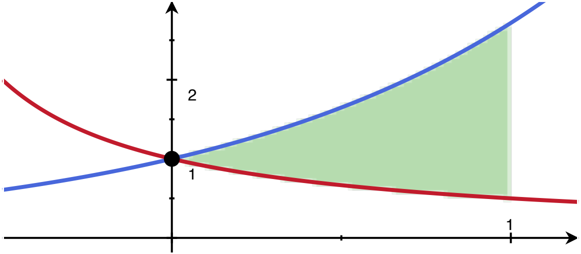
Intersections: where does $x^2 = x^3$? $\boxed{x = 0 \text{ or } 1}$

So $$\text{Area} = \int_0^1 x^2 - x^3 dx = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_{x=0}^{1} = \boxed{\left(\tfrac{1}{3} - \tfrac{1}{4}\right) - 0} > 0 \checkmark$$

Example

Find the area of the region between $y = e^x$ and $y = 1/(1 + x)$ on the interval $[0, 1]$.



1. Check for intersection points (verify algebraically that $x = 0$ is the only intersection by setting $e^x = \frac{1}{x+1}$).
2. Decide which function is on top $(f(x))$ and which function is on bottom $(g(x))$.
3. Calculate $\int_0^1 f(x) - g(x) dx$.

Check: What if you get a negative answer?

Example

Find the area of the region bounded by $y = x^2 - 2x$ and $y = 4 - x^2$.

1. Check for intersection points (Solve $x^2 - 2x = 4 - x^2$). There will be two, a and b; this is where the functions cross.
2. Between this two points, which function is on top ($f(x)$) and which function is on bottom ($g(x)$).
3. Calculate $\int_a^b f(x) - g(x)dx$.

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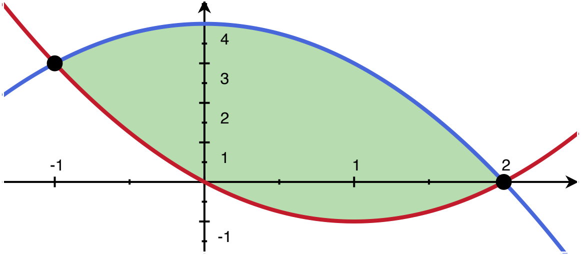

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Find the area of the region bounded by the two curves $y = x^3 - 9x$ and $y = 9 - x^2$.

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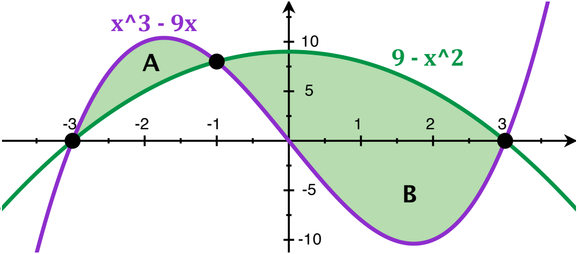

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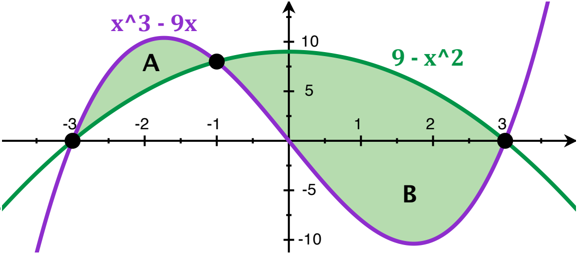

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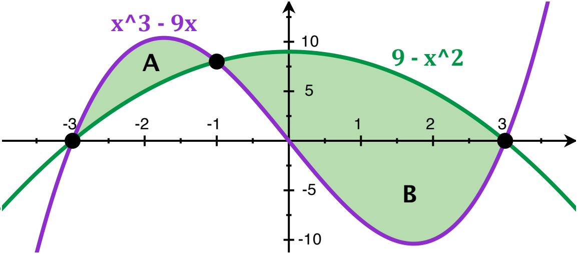

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$$\text{Area A} = \int_{-3}^{-1} (x^3 - 9x) - (9 - x^2) dx = \int_{-3}^{-1} x^3 + x^2 - 9x - 9 \, dx$$

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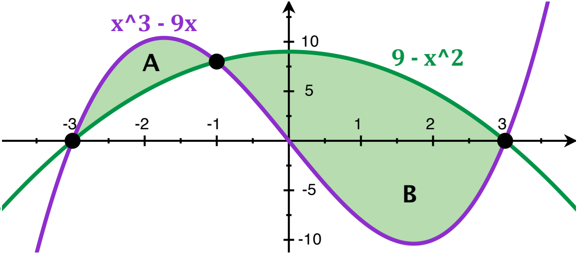

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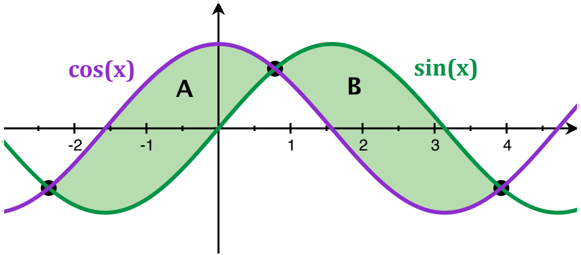
$$\text{Area B} = \int_{-1}^{3} (9 - x^2) - (x^3 - 9x)\,dx = -\int_{-1}^{3} x^3 + x^2 - 9x - 9\;dx$$

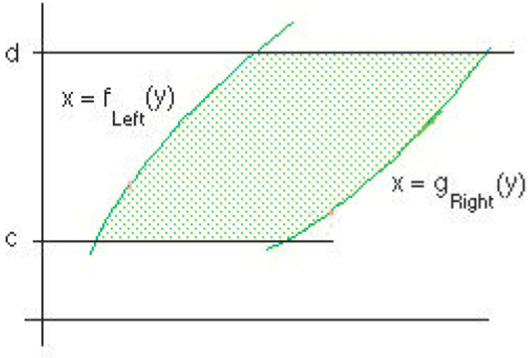
Example

Find the area between $\sin x$ and $\cos x$ on $[-3\pi/4, 5\pi/4]$.



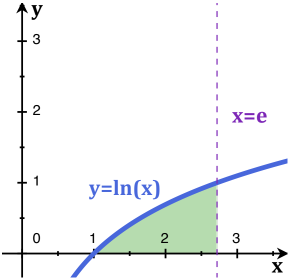
Functions of y

We could just as well consider two functions of y, say, $x = f_{Left}(y)$ and $x = g_{Right}(y)$ defined on the interval $[c, d]$.



Area Between the Two Curves

Find the area under the graph of $y = \ln x$ and above the interval $[1, e]$ on the x-axis.



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