The FUNDAMENTAL Theorem of Calculus (yay!)

11/11/11
(also yay!)

Warm-up

Suppose a particle is traveling at velocity $v(t) = t^2$ from $t = 1$ to $t = 2$. if the particle starts at $y(0) = y_0$,

1. what is the function $y(t)$ which gives the particles position as a function of time (will have a y_0 in it)?
2. how far does the particle travel from $t = 1$ to $t = 2$?

Compare your answer to the upper and lower estimates of the area under the curve $f(x) = x^2$ from $x = 1$ to $x = 2$:

Upper $U(f,P)$

$$\sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) * \left(\frac{1}{n}\right)$$

Lower $L(f,P)$

$$\sum_{i=0}^{n-1} \left(1 + \frac{i}{n}\right) * \left(\frac{1}{n}\right)$$

n	$U(f,P)$	$L(f,P)$
10	2.485	2.185
100	2.34835	2.31835
1000	2.33483	2.33183

The Fundamental Theorem of Calculus

Theorem (the baby case)

If $F(x)$ is any function satisfying $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_{x=a}^{b} = F(b) - F(a)$$

Q. What is $\int_1^2 x^2 dx$?

A. $F(x) = \frac{x^3}{3}$
So

$$\int_1^2 x^2 dx = \frac{x^3}{3}\Bigg|_{x=1}^{2} = \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right)$$

$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3} \approx 2.333} \qquad \text{(same answer!)}$$

Examples

Use the fundamental theorem of calculus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

to calculate

1. $\int_2^3 3x\ dx$
2. $\int_{-1}^1 x^3\ dx$
3. $\int_0^\pi \sin(x)\ dx$
4. $\int_\pi^0 \sin(x)\ dx$

The Fundamental Theorem of Calculus

Theorem (the big case)

If $F(x)$ is any function satisfying $\frac{d}{dt}F(t) = f(t)$, then

$$\int_a (x)^b (x) f(t) dt = F(t)\Big|_{t=a(x)}^{b(x)} = F(b(x)) - F(a(x))$$

Q. What is $\displaystyle\int_{\sin(x)}^{\ln(x)} t^2 dt$?

A. $F(t) = \dfrac{1}{3}t^3$.

So

$$\int_{\sin(x)}^{\ln(x)} t^2 dt = \frac{1}{3}t^3\Big|_{t=\sin(x)}^{\ln(x)} = \left(\frac{1}{3}(\ln(x))^3\right) - \left(\frac{1}{3}(\sin(x))^3\right).$$

Examples

Use the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(t)dt = F(b(x)) - F(a(x))$$

to calculate

1. $\int_{\sin(x)}^{\cos(x)} 3t \; dt$
2. $\int_{x+1}^{5x^2-3} t^3 \; dt$
3. $\int_{\arccos(x)}^{0} \sin(t) \; dt$

For reference, we calculated $\int_{a(x)}^{\ln(x)} f(t)\, dt$ where

$$f(t) = t^2 \qquad a(x) = \sin(x) \qquad b(x) = \ln(x).$$

Notice:

$$\frac{d}{dx}\left(\frac{1}{3}(\ln(x))^3 - \frac{1}{3}(\sin(x))^3\right) = \frac{1}{x}(\ln(x))^2 - \cos(x)(\sin(x))^2$$
$$= b'(x)f(b(x)) - a'(x)f(a(x)).$$

In general:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\, dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

(Don't even have to know $F(t)$!)

Why?

Example: Calculate $\dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt$.

Answer: We can't even calculate $\displaystyle\int e^{t^2}\,dt$!

(There is no elementary function $F(t)$ which satisfies $F'(t) = e^{t^2}$)

But we know $\displaystyle\int e^{t^2}\,dt$ is a function. Call it $F(t)$.

So $\displaystyle\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt = F(\sin(x)) - F(\tan(x))$.

Therefore

$$\begin{aligned}
\frac{d}{dx}\int_{\tan(x)}^{\sin(x)} e^{t^2}\,dt &= \frac{d}{dx}\left(F(\sin(x)) - F(\tan(x))\right) \\
&= \cos(x)F'(\sin(x)) - \sec^2(x)F'(\tan(x)) \\
&= \cos(x)f(\sin(x)) - \sec^2(x)f(\tan(x)) \\
&= \cos(x)e^{(\sin(x))^2} - \sec^2(x)e^{(\tan(x))^2}
\end{aligned}$$

Part 2: Integration by substitution

Warmup

Fill in the blank:

1. Since $\dfrac{d}{dx}\cos(x^2+1) = \rule{4cm}{0.4pt}$,

   $$\text{so} \int \rule{4cm}{0.4pt} \, dx = \cos(x^2+1) + C.$$

2. Since $\dfrac{d}{dx}\ln|\cos(x)| = \dfrac{}{\rule{3cm}{0.4pt}}$,

   $$\text{so} \int \frac{}{\rule{3cm}{0.4pt}} \, dx = \ln|\cos(x)| + C.$$

(Example: $\frac{d}{dx}x^3 dx = 3x^2$, so $\int 3x^2 dx = x^3 + C$.)

Undoing chain rule

In general:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x),$$

so $$\int f'(g(x)) * g'(x)dx = f(g(x)) + C.$$

Example: Calculate the (extremely suggestively written) integral

$$\int \cos(x^3 + 5x - 10) * (3x^2 + 5 * 1 + 0)dx$$

Less obvious chain rules.

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx:

Examples:

$$\int \frac{\cos(x)}{\sin(x)+1} dx = \int \frac{1}{\sin(x)+1} * \cos(x)\ dx$$

$$\int x\sqrt{x^2+1} dx = \int \frac{1}{2}\sqrt{x^2+1} * 2x\ dx$$

$$\int \frac{\cos(\sqrt{x})}{2\sqrt{x}} dx = \int \cos(\sqrt{x}) * \frac{1}{2\sqrt{x}}\ dx$$

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

$$\text{Example:} \quad \int \frac{\cos(x)}{\sin(x)+1}dx$$

Let $u = g(x)$.	Let $u = \sin(x) + 1$
Calculate du.	$\frac{du}{dx} = \cos(x)$ so $du = \cos(x)dx$
Clear out all of the x's, replacing them with u's.	$\int \frac{1}{u} du$
Calculate the new integral.	$\int \frac{1}{u} du = \ln\|u\| + C$
Substitute back into x's.	$\ln\|u\| + C = \ln\|\sin(x)+1\| + C$

Check $\frac{d}{dx} \ln|\sin(x)+1| + C = \frac{1}{\sin(x)+1} * \cos(x)$ ✓

Method of Substitution

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

Example: $\int x\sqrt{x^2+1}\,dx$

Let $u = g(x)$.	Let $u = x^2 + 1$
Calculate $c * du$.	$\frac{du}{dx} = 2x$ so $\frac{1}{2}du = x\ dx$
Clear out all of the x's, replacing them with u's.	$\int \sqrt{u} * \frac{1}{2}du$
Calculate the new integral.	$\frac{1}{2}\int u^{1/2}du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$
Substitute back into x's.	$= \frac{1}{3}(x^2+1)^{3/2} + C$

Check $\frac{d}{dx}\frac{1}{3}(x^2+1)^{3/2} + C = \frac{1}{3}\frac{3}{2}(x^2+1)^{1/2} * 2x$✓

Give it a try:

Look for a buried function $g(x)$ and it's derivative $g'(x)$ which can be paired with dx.

$$\text{Example:} \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

Let $u = g(x)$.

Calculate $c * du$.

Clear out all of the x's,
replacing them with u's.

Calculate the new integral.

Substitute back into x's.