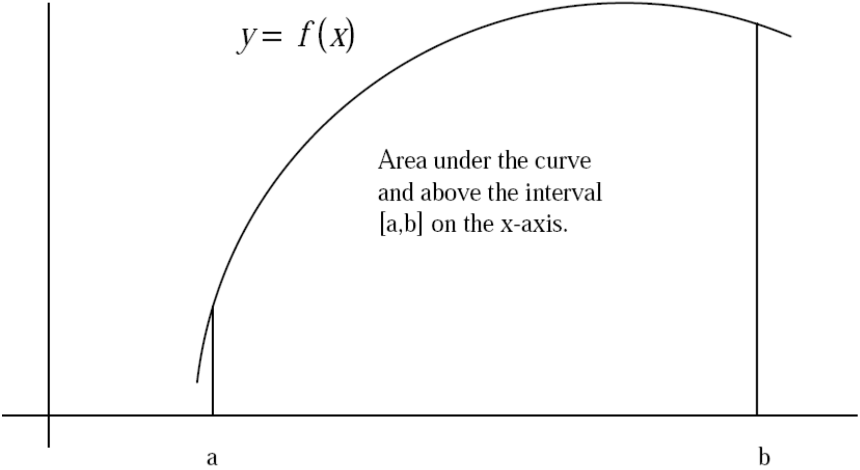
The Definite Integral

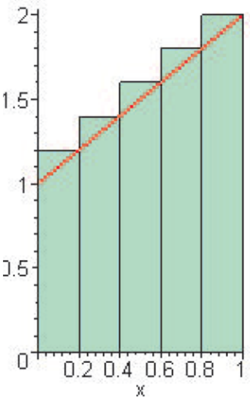
11/09/2011

The Area Problem

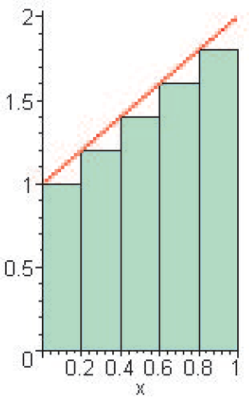


Upper and Lower Sums

Suppose we want to use rectangles to approximate the area under the graph of $y = x + 1$ on the interval $[0, 1]$.



Upper Riemann Sum



Lower Riemann Sum

$$31/20 \quad > \quad 1.5 \quad > \quad 29/20$$

As you take more and more smaller and smaller rectangles, if f is nice, both of these will approach the real area.

n	U	L
100	1.505000000	1.495000000
150	1.503333333	1.496666667
200	1.502500000	1.497500000
300	1.501666667	1.498333333
500	1.501000000	1.499000000

In general: finding the Area Under a Curve

1. Let $y = f(x)$ be given and defined on an interval $[a, b]$. Subdivide the interval $[a, b]$ into n pieces. Label the endpoints:

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b.$$

   Define $P = \{x_0, x_1, x_2, \ldots .x_n\}$.

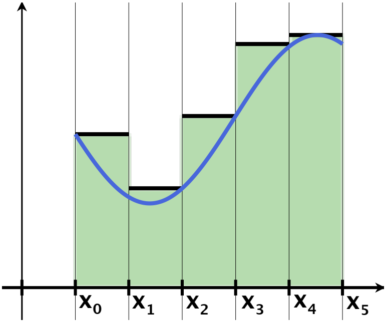
2. Let $\Delta x_i = x_i - x_{i-1}$ be the width of the i^{th} interval, $1 \leq i \leq n$.
3. Form the Upper Riemann Sum $U(f, P)$: let M_i be the *maximum* value of the function on that i^{th} interval, so

$$U(f, P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n.$$

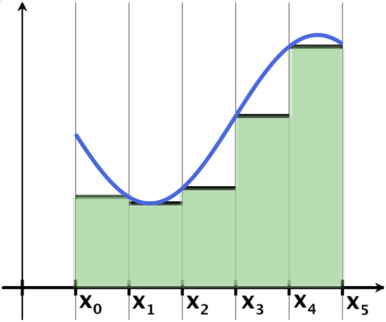
4. Form the Lower Riemann Sum $L(f, P)$: let m_i be the *minimum* value of the function on that i^{th} interval, so

$$L(f, P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n.$$

5. Take the limit as $n \to \infty$ and the maximum $\Delta x_i \to 0$.



$U(f,P)$



$L(f,P)$

Sigma Notation

If m and n are integers with $m \leq n$, and if f is a function defined on the integers from m to n, then the symbol $\sum_{i=m}^{n} f(i)$, called sigma notation, is means

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n)$$

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Examples:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$\sum_{i=1}^{n} \sin(i) = \sin(1) + \sin(2) + \sin(3) + \cdots + \sin(n)$$

$$\sum_{i=0}^{n-1} x^i = x^0 + x + x^2 + x^2 + x^3 + x^4 + \cdots + x^{n-1}$$

Sigma Notation

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$$\sum_{i=0}^{n-1} x^i = 1 + x + x^2 + x^2 + x^3 + x^4 + \cdots + x^{n-1}$$

Simplifying long sums

① Distribute and simplify:
(hint: first distribute 1 and then $-x$. group "like terms")

(a.) $(1-x)(1+x) = 1-x^2$
(b.) $(1-x)(1+x+x^2) = 1-x^3$
(c.) $(1-x)(1+x+x^2+x^3) = 1-x^4$
(d.) $(1-x)(1+x+x^2+x^3+x^4) = 1-x^5$

② In (a.)-(d.), solve for $\sum_{i=0}^{n-1} x^i$
(in (a.) $n=2$, in (b.) $n=3$, in (c.) $n=4$, in (d.) $n=5$)
What did you have to assume to do this?

③ Write a general simple (compact) formula for $\sum_{i=0}^{n-1} x^i$.

④ Simplify $\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5\right)$ (hint: $x = \frac{1}{3}$)

② (a.) $1+x = \frac{1-x^2}{1-x}$

(b.) $1+x+x^2 = \frac{1-x^3}{1-x}$ ← 3

(c.) $1+x+x^2+x^3 = \frac{1-x^4}{1-x}$

(d.) $1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$

③ $1+x+x^2+\dots+x^{n-1} = \frac{1-x^n}{1-x}$

$$\sum_{i=0}^{5} \left(\frac{1}{3}\right)^i$$

$$= 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5$$

$$n = 6 \quad , \quad x = \frac{1}{3}$$

$$= \frac{1 - (1/3)^6}{\underbrace{1 - (1/3)}_{2/3}} = \frac{3}{2}\left(\underbrace{1 - (1/3)^6}_{??}\right)$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^6}\right)$$

$$= \frac{3}{2}\left(\frac{3^6 - 1}{3^6}\right)$$

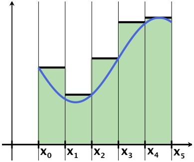
$$= \frac{1}{2}\left(\frac{3^6 - 1}{3^5}\right)$$

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

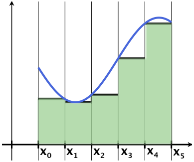
The Area Problem Revisited

$$
\begin{aligned}
U(f, P) &= \sum_{i=1}^{n} M_i \Delta x_i \\
L(f, P) &= \sum_{i=1}^{n} m_i \Delta x_i,
\end{aligned}
$$

where M_i and m_i are, respectively, the maximum and minimum values of f on the ith subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$.



$U(f, P)$



$L(f, P)$

$n = 5$

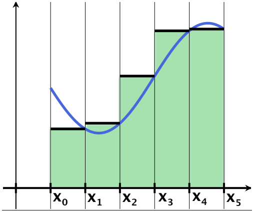
Riemann Sums

Given a partition P of $[a,b]$, $P = \{a = x_0, x_1, x_3, \ldots, x_n = b\}$, and $\Delta x_i = x_i - x_{i-1}$ the width of the ith subinterval, $1 \leq i \leq n$;

Let f be defined on $[a,b]$.

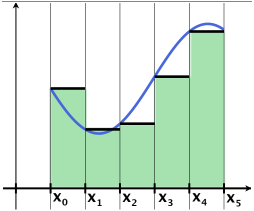
Then the Right Riemann Sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x_i,$$



and the Left Riemann Sum is

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i.$$



The Definite Integral

Let P be a partition of the interval $[a, b]$, $P = \{x_0, x_1, x_2, ..., x_n\}$ with $a = x_0 \leq x_1 \leq x_2 \ldots x_n = b$.

Let $\Delta x_i = x_i - x_{i+1}$ be the width of the ith subinterval, $1 \leq i \leq n$. Let f be a function defined on $[a, b]$.

We say that f is Riemann integrable on $[a, b]$ if there exists a number A such that $L(f, P) \leq A \leq U(f, P)$ for all partitions of $[a, b]$. We write the number as

$$A = \int_a^b f(x)dx$$

and call it the definite integral of f over $[a, b]$.

Theorem

If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Theorem

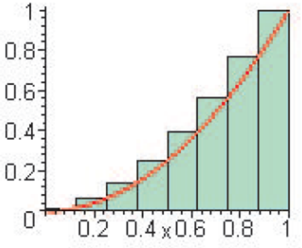
If f is Riemann integrable on $[a, b]$, then

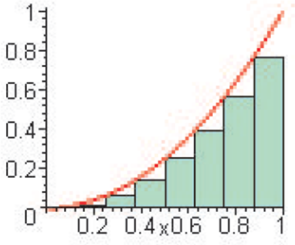
$$\int_a^b f(x)dx = \lim_{\substack{n \to \infty \\ ||P|| \to 0}} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where c_i is any point in the interval $[x_{i-1}, x_i]$ and $||P||$ is the maximum length of the Δx_i.

Example

Use an Upper Riemann Sum and a Lower Riemann Sum, first with 8, then with 100 subintervals of equal length to approximate the area under the graph of $y = f(x) = x^2$ on the interval $[0, 1]$.





$$U(f,P) = \sum_{i=1}^{8} \left(\frac{i}{8}\right)^2 \frac{1}{8}$$

b/c $x_0 = 0$, $x_1 = \frac{1}{8}$, $x_2 = \frac{2}{8}$ ... $x_i = \frac{i}{8}$

$$L(f,P) = \sum_{i=0}^{7} \left(\frac{i}{8}\right)^2 \frac{1}{8} \quad .$$

↑

not x^i

is

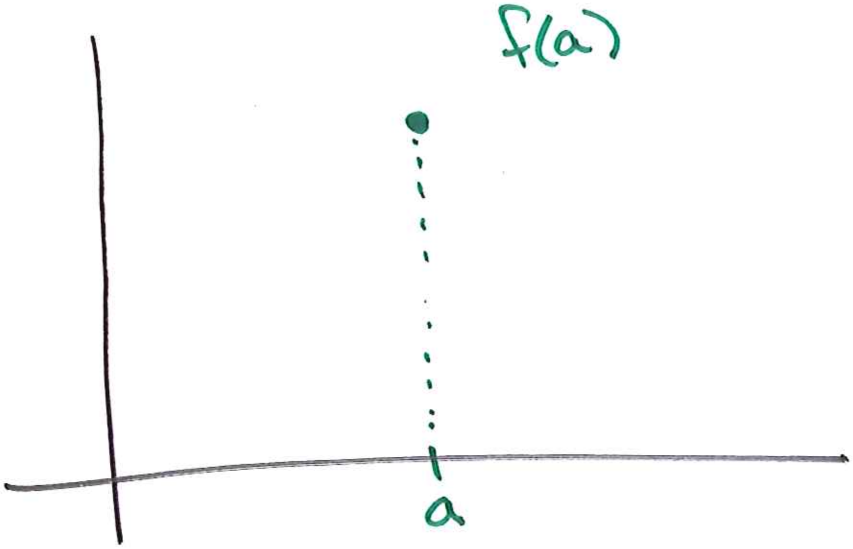
$(i)^x$

Properties of the Definite Integral

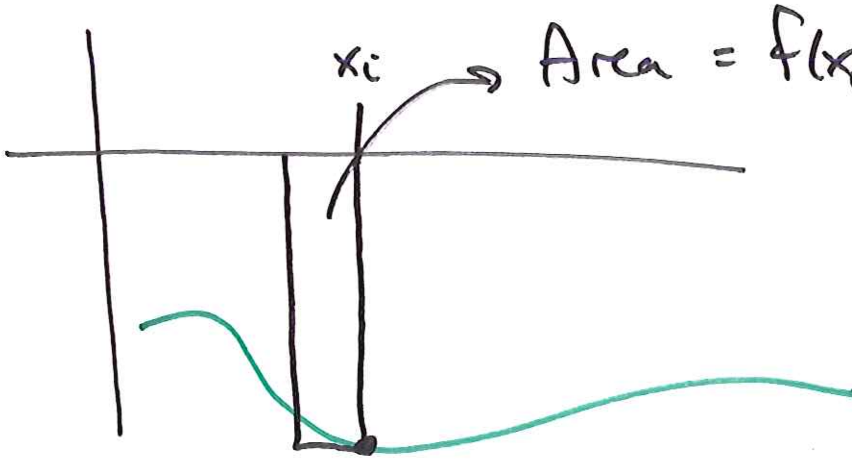
1. $\displaystyle\int_a^a f(x)dx = 0.$

2. If f is integrable and
   (a) $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the area of the region under the graph of f and above the interval $[a, b]$;
   (b) $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the **negative** of the area of the region between the interval $[a, b]$ and the graph of f.

3. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx.$

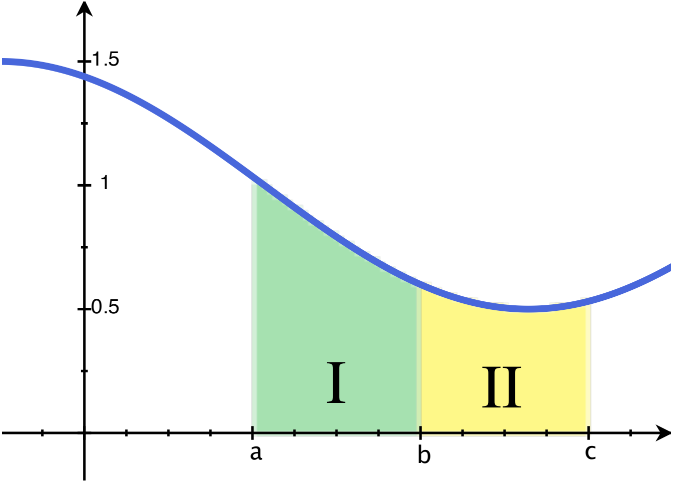


$A = 0.$



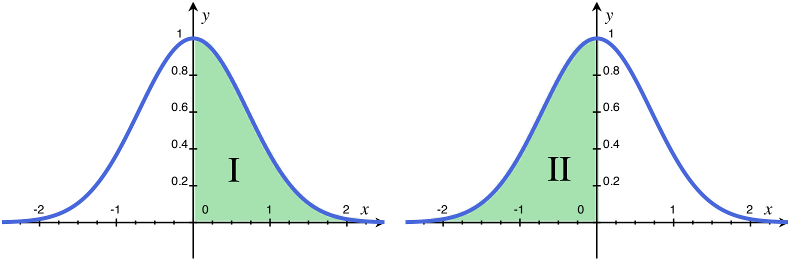
$\text{Area} = f(x_i)\Delta X < 0.$

4. If $a < b < c$, $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$



5. If f is an **even** function, then

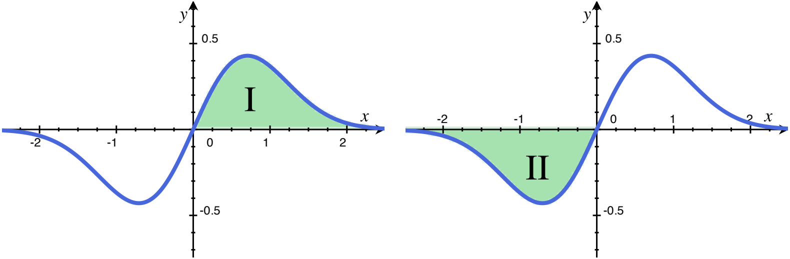
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



Area I = Area II

6. If f is an **odd** function, then

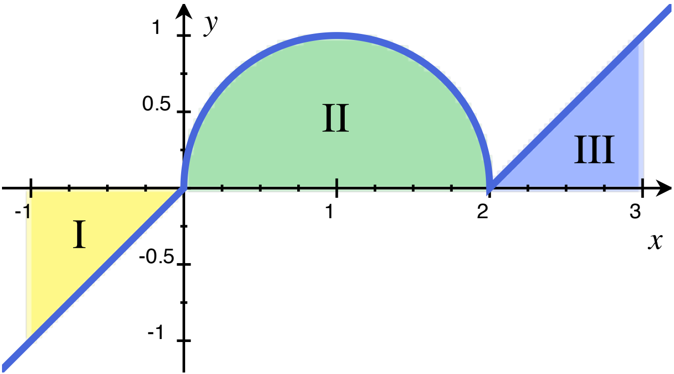
$$\int_{-a}^{a} f(x)dx = 0.$$



Area I = Area II

Example

$$\text{If } f(x) = \begin{cases} x, & x < 0, \\ \sqrt{1-(x-1)^2}, & 0 \geq x \leq 2, \\ x-2, & x \geq 2, \end{cases} \text{ what is } \int_{-1}^{3} f(x)dx?$$



$$A(I) = \frac{1}{2}$$

$$\int_{-1}^{0} f(x)\,dx = -\frac{1}{2}$$

$$A(II) = \frac{\pi 1^2}{2} = \frac{\pi}{2}$$

$$\int_{0}^{2} f(x)\,dx = \frac{\pi}{2}$$

$$A(III) = \frac{1}{2}$$

$$\int_{2}^{3} f\,dx = 1/2$$

$$\int_{-1}^{3} f(x)\,dx = \int_{-1}^{0} f\,dx + \int_{0}^{2} f\,dx + \int_{2}^{3} f\,dx$$

$$= -\frac{1}{2} + \frac{\pi}{2} + \frac{1}{2} = \frac{\pi}{2}.$$

Mean Value Theorem for Definite Integrals

Theorem

Let f be continuous on the interval $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c).$$

Definition

The *average value* of a continuous function on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx.$$