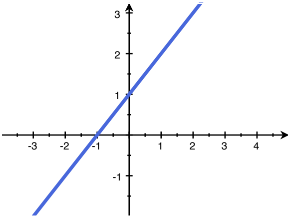
Curve Sketching

11/2/2011

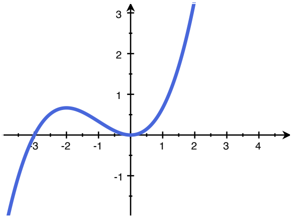
Warm up

Below are pictured six functions: f, f', f'', g, g', and g''. Pick out the two functions that could be f and g, and match them to their first and second derivatives, respectively.

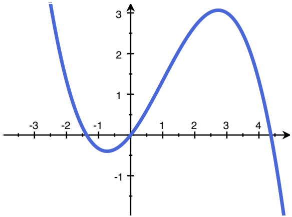
(a)



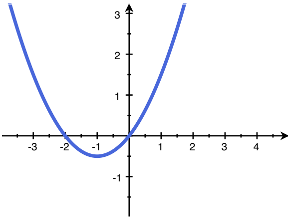
(b)



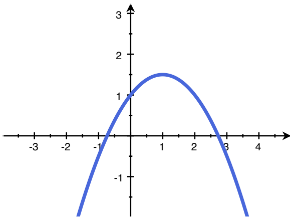
(c)



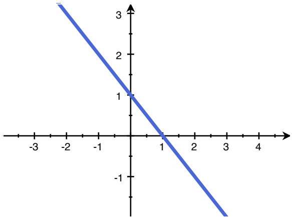
(e)



(f)



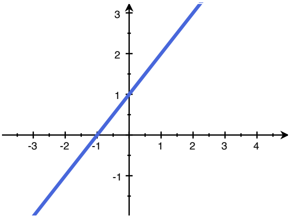
(g)



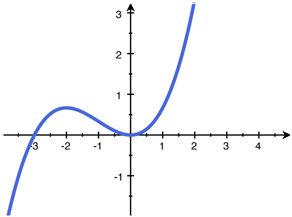
Warm up

Below are pictured six functions: f, f', f'', g, g', and g''. Pick out the two functions that could be f and g, and match them to their first and second derivatives, respectively.

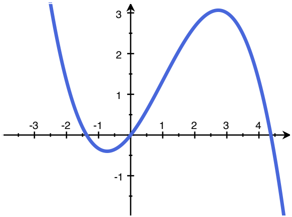
f''



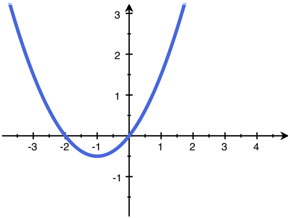
f



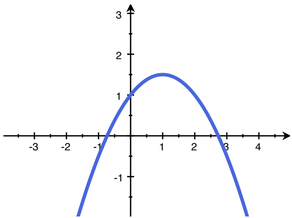
g



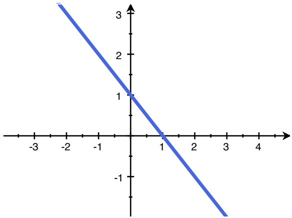
f'



g'



g''



Review: Monotonicity of functions on intervals

Suppose that the function f is defined on an interval I, and let x_1 and x_2 denote points in I:

1. f is increasing on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

2. f is decreasing on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

3. f is constant on I if $f(x_1) = f(x_2)$ for any x_1, x_2 in I.

Review: Testing monotonicity via derivatives

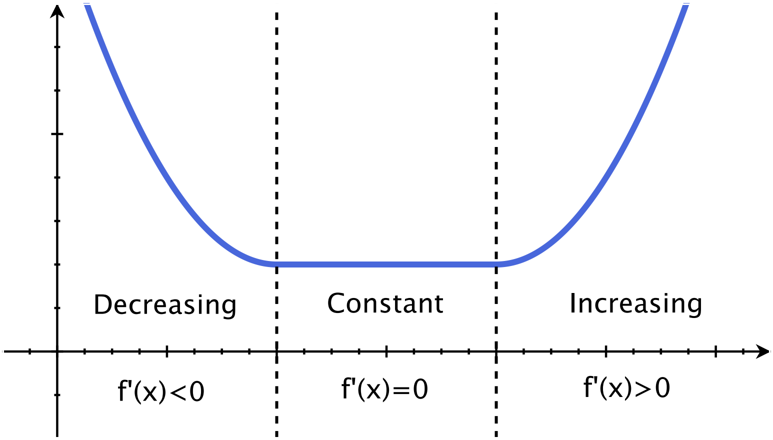
Recall: The derivative function $f'(x)$ tells us the slope of the tangent line to the graph of the function f at the point $(x, f(x))$.

Theorem (Increasing/Decreasing Test)

Suppose that f is continuous on $[a, b]$ and differentiable on the open interval (a, b) . Then

1. *If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.*
3. *If $f'(x) = 0$ for every x in (a, b), then f is constant on $[a, b]$.*

What it looks like:



Theorem (Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then

1. *there is a point c_1 in the interval where f assumes it maximum value, i.e. $f(x) \leq f(c_1)$ for all x in $[a, b]$, and*
2. *there is a point c_2 in the interval where f assumes its minimal value, i.e. $f(x) \geq f(c_2)$ for all x in $[a, b]$.*

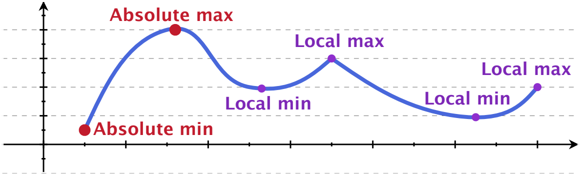
Finding minima and maxima is all about *optimizing* a function. So how do we find these values?

Finding Extreme Values with Derivatives

Theorem

If f is continuous in an open interval (a, b) and achieves a maximum (or minimum) value at a point c in (a, b) where $f'(c)$ exists, then either $f'(c)$ is not defined or $f'(c) = 0$.

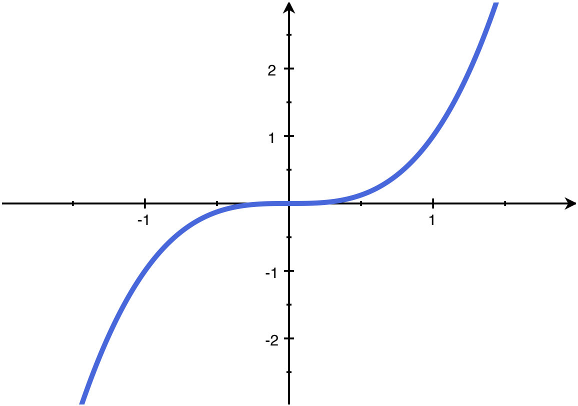
Big Idea: if $f'(c)$ exists, and is not equal to 0, then $f(x)$ is either increasing or decreasing on both sides of c, so $f(c)$ could not be a min or a max.



Definition: A point $x = c$ where $f'(c) = 0$ or where $f'(c)$ doesn't exist is called a **critical point**. If $f'(c)$ is undefined, c is also called a **singular point**.

Warning: Not all critical points are local minima or maxima:

Example: If $f(x) = x^3$, then $f'(x) = 3x^2$, and so $f'(0) = 0$:



Strategy for closed bounded intervals

1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Strategy for closed bounded intervals

1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

Strategy for closed bounded intervals

1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$$

Strategy for closed bounded intervals

1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

So $f'(x) = 0$ if $x = -1$ or 1.

Strategy for closed bounded intervals

1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$$

So $f'(x) = 0$ if $x = -1$ or $\boxed{1}$.

Strategy for closed bounded intervals

1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$$

So $f'(x) = 0$ if $x = -1$ or $\boxed{1}$.

x	$f(x)$	
1	-2	critical points
0	0	end points
2	2	

Strategy for closed bounded intervals

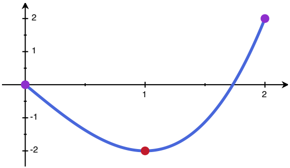
1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$$

So $f'(x) = 0$ if $x = -1$ or $\boxed{1}$.

x	$f(x)$	
1	-2	critical points
0	0	end points
2	2	

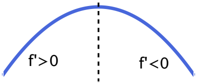


First Derivative Test

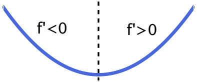
Finding local extrema can be useful for sketching curves.

Let c be a critical/singular point of of a function $y = f(x)$ that is continuous on an open interval $I = (a, b)$ containing c. If f is differentiable on the interval (except possibly at the singular point $x = c$) then the value $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a local maximum.



2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a local minimum.



3. If $f'(x)$ doesn't change sign, then it's neither a min or a max.

Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

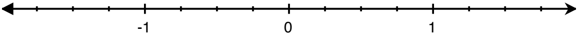
$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$

Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$

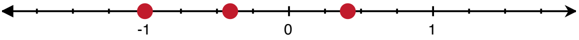


Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$

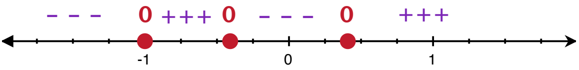


Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$

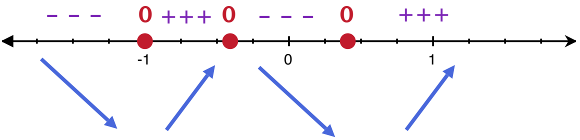


Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$

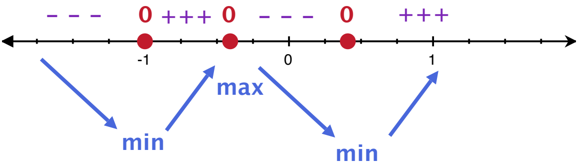


Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$

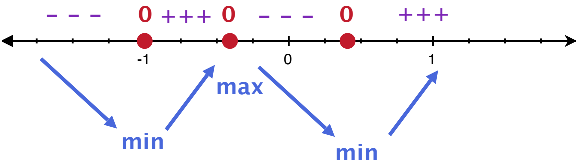


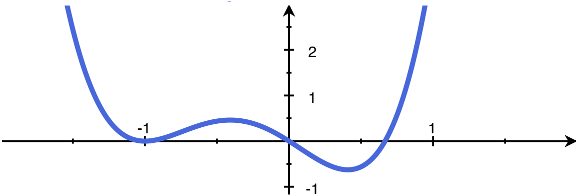
Example

Find the local extrema of $f(x) = 3x^4 + 4x^3 - x^2 - 2x$ over the whole real line.

Calculate $f'(x)$:

$$f'(x) = 12x^3 + 12x^2 - 2x - 2 = 12(x+1)(x - 1/\sqrt{6})(x + 1/\sqrt{6})$$





Example

Find the local extrema of $f(x) = \dfrac{x^4 + 1}{x^2}$ over the whole real line.

[Hint: find a common denominator after taking a derivative.]

$$f = x^2 + \frac{1}{x^2} \rightarrow x^{-2}$$

$$f' = 2x - 2x^{-3} \rightarrow \frac{2}{x^3}$$

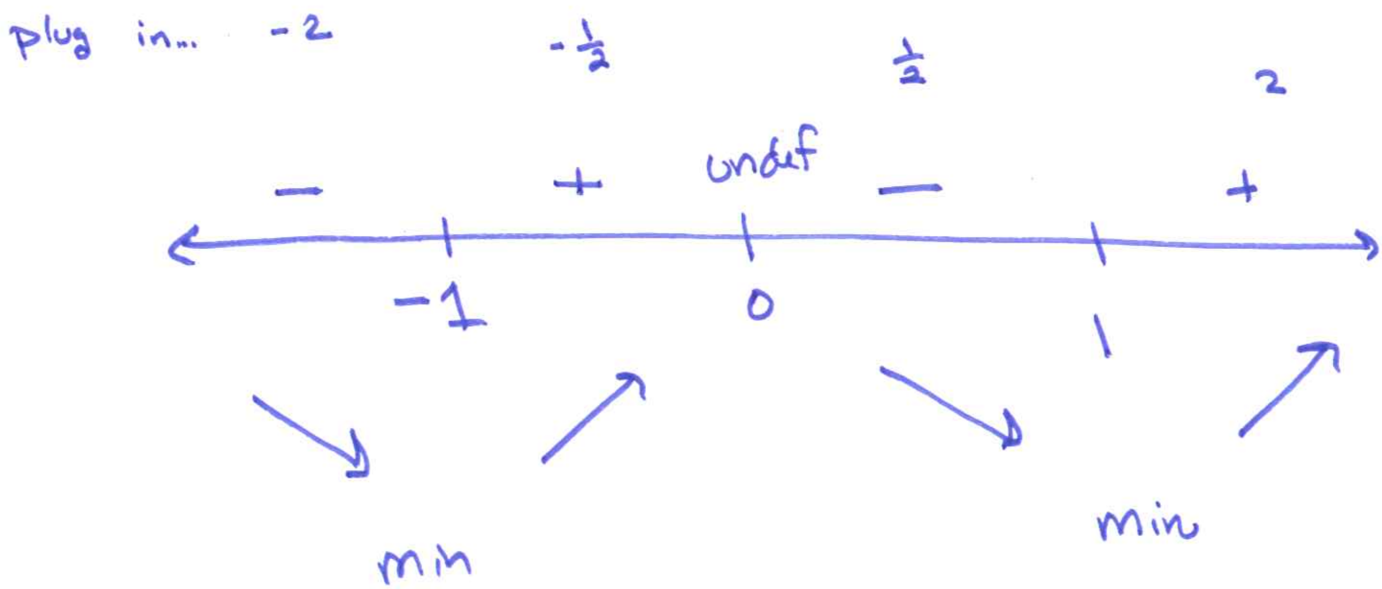
$$= \frac{2x^4 - 2}{x^3} = 2\,\frac{x^4 - 1}{x^3}$$

undefined @ $x = 0$

$f'(x) = 0$ when $x^4 - 1 = 0$

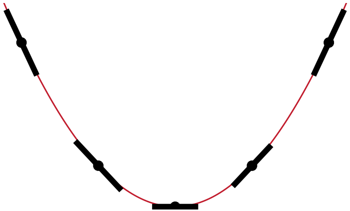
$(x^2+1)(x+1)(x-1) =$

so critical points: $\boxed{x = 0, 1, -1}$

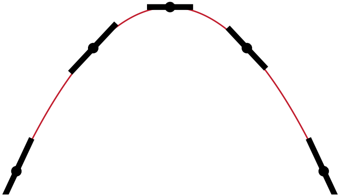


Concavity

Q. How can we measure when a function is concave up or down?



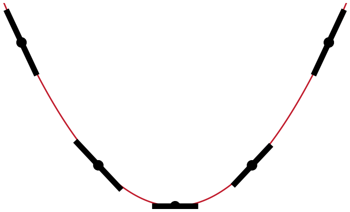
Concave up
$f'(x)$ is increasing



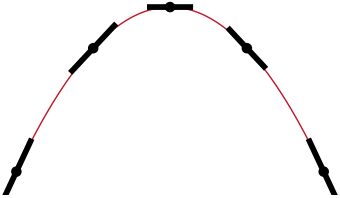
Concave down
$f'(x)$ is decreasing

Concavity

Q. How can we measure when a function is concave up or down?



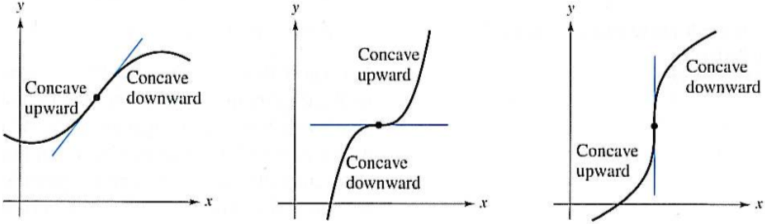
Concave up
$f'(x)$ is increasing
$\boxed{f''(x) > 0}$



Concave down
$f'(x)$ is decreasing
$\boxed{f''(x) < 0}$

Concavity and Inflection Points

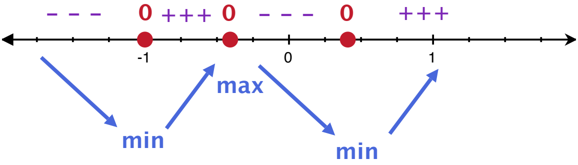
Definition: The function f has an **inflection point** at the point $x = c$ if $f'(c)$ exists and the concavity changes at $x = c$ from up to down or vice versa.



Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

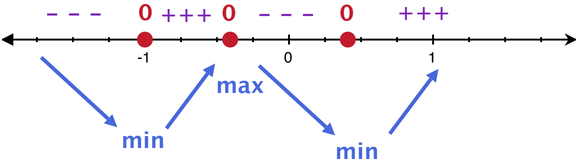
We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.



Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.



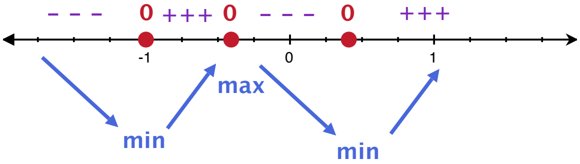
So

$$f''(x) = 36x^2 + 24x - 2 = (6x - (\sqrt{6} - 2))(6x + \sqrt{6} + 2)$$

Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

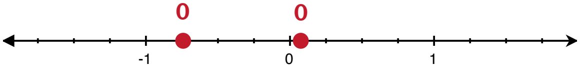
Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.



So

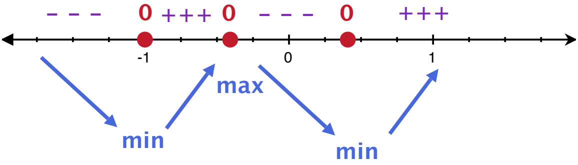
$$f''(x) = 36x^2 + 24x - 2 = (6x - (\sqrt{6} - 2))(6x + \sqrt{6} + 2)$$



Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

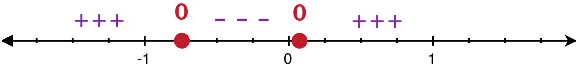
Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.



So

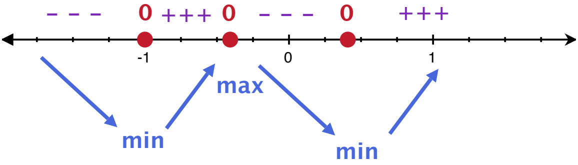
$$f''(x) = 36x^2 + 24x - 2 = (6x - (\sqrt{6} - 2))(6x + \sqrt{6} + 2)$$



Back to the example $f(x) = 3x^4 + 4x^3 - x^2 - 2x$

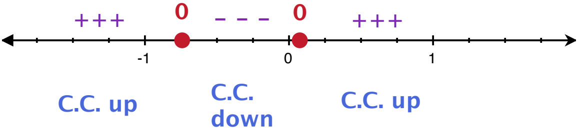
Find the inflection points of $f(x)$, and where $f(x)$ is concave up or down.

We calculated $f'(x) = 12x^3 + 12x^2 - 2x - 2$.

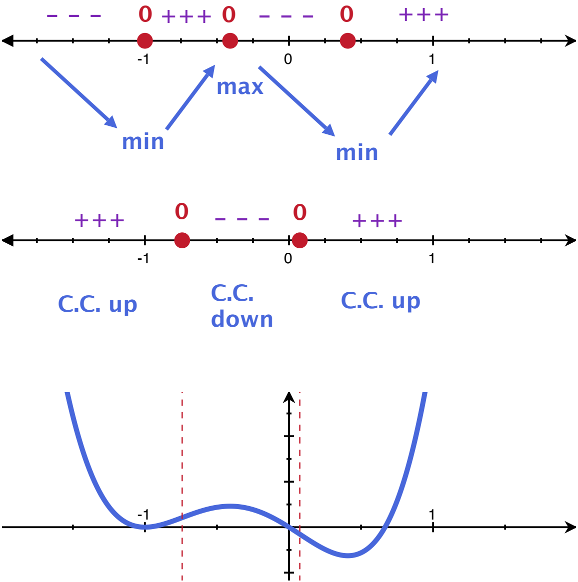


So

$$f''(x) = 36x^2 + 24x - 2 = (6x - (\sqrt{6} - 2))(6x + \sqrt{6} + 2)$$



Putting it together

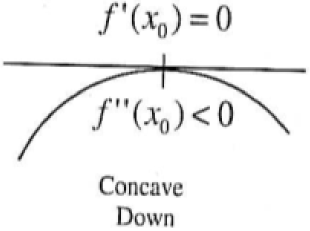


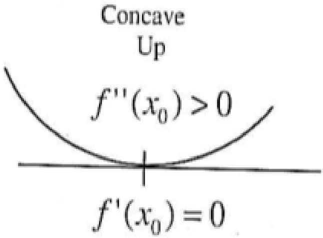
The second derivative test

Theorem

Let f be a function whose second derivative exists on an interval I containing x_0.

1. *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum.*
2. *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ is a local maximum.*
3. *If $f'(x_0) = 0$ and $f''(x_0) <= 0$, then the test* **fails**, *use the first derivative test to decide.*





Sketch graphs of the following functions:

1. $f(x) = -3x^5 + 5x^3$.
2. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

Instructions:

Step 1 Find any places where $f(x)$ is 0 or undefined.

Step 2 Calculate $f'(x)$ and find critical/singular points.

Step 3 Classify where $f'(x)$ is positive/negative, and therefore where $f(x)$ is increasing/decreasing.

Step 4 Calculate $f''(x)$, and find where it's 0 or undefined.

Step 5 Classify where $f''(x)$ is positive/negative, and therefore where $f(x)$ is concave up/down.

Step 6 Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to see what the tails are doing.

Hint for 2: Always simplify as a fraction of polynomials after taking a derivative.

(1)

$$f(x) = -3x^5 + 5x^3 = x^3(-3x^2+5) = -3(x^3)(x^2 - \tfrac{5}{3})$$

$f = 0$ @ $x = 0$, $x = \pm\sqrt{5/3}$

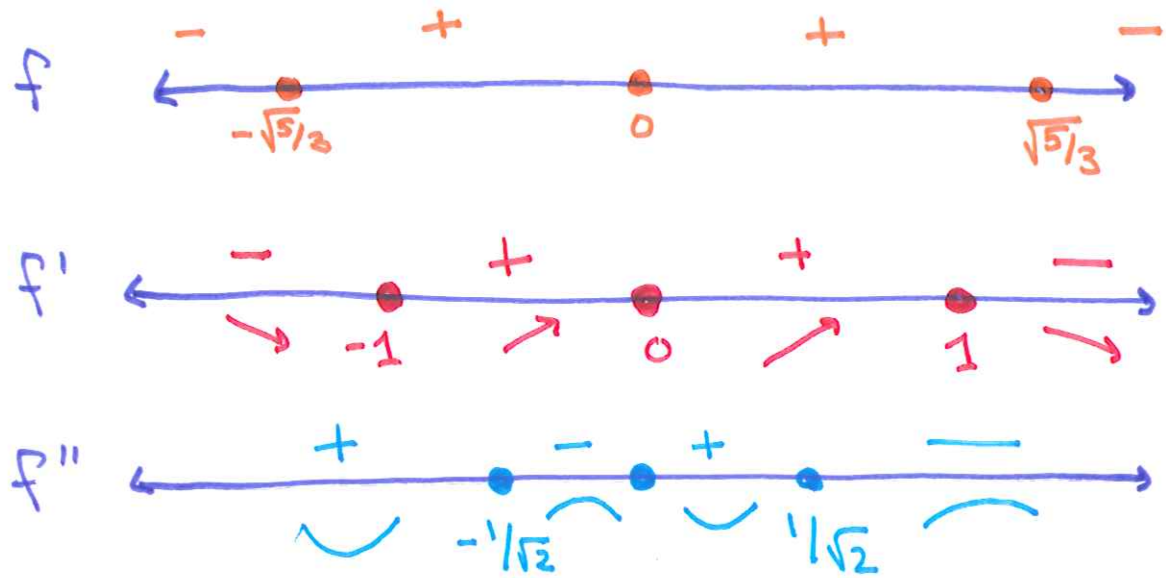
$$f'(x) = -15x^4 + 15x^2 = -15(x^4 - x^2)$$

$$= -15x^2(x+1)(x-1)$$

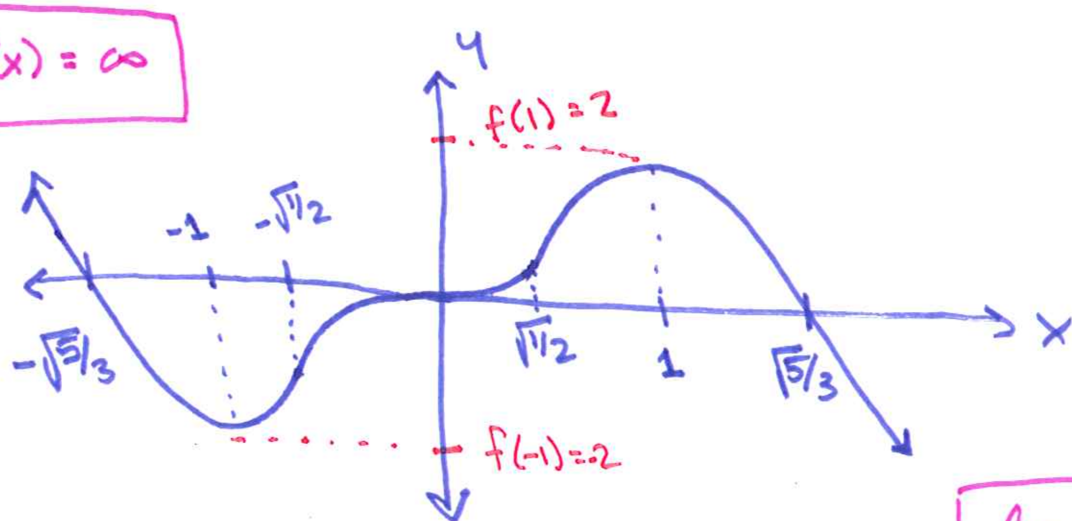
$f' = 0$ @ $x = 0, \pm 1$

$$f''(x) = -60x^3 + 30x = -60(x)(x^2 - \tfrac{1}{2})$$

$f'' = 0$ @ $x = 0$, $x = \pm\sqrt{1/2}$



$\lim_{x \to -\infty} f(x) = \infty$



$\lim_{x \to \infty} f(x) = -\infty$

(2) $f(x) = \dfrac{x^2-1}{x^2+1} = \dfrac{(x+1)(x-1)}{x^2+1}$

so $f(x) = 0$ when $\boxed{x = \pm 1}$
(and is defined everywhere since $x^2+1 > 0$)

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

$f'(x) = 0$ when $\boxed{x=0}$
(and is defined everywhere)

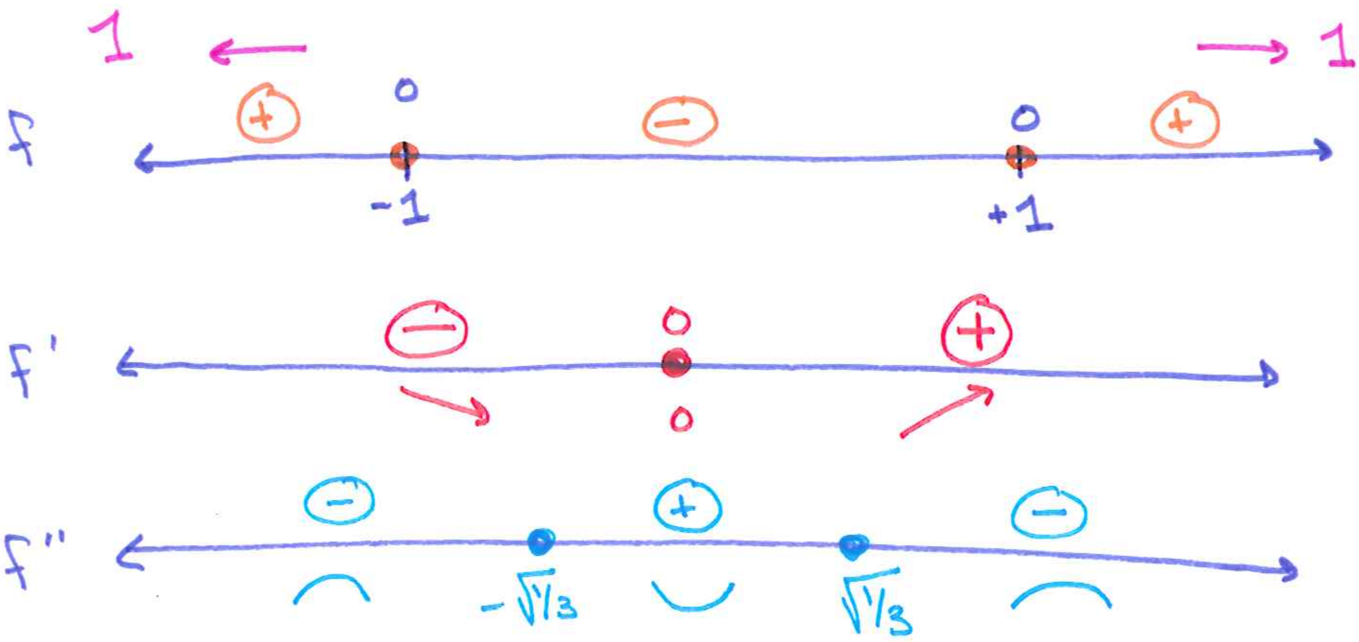
$$f''(x) = \frac{4(x^2+1)^2 - 4x \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$$

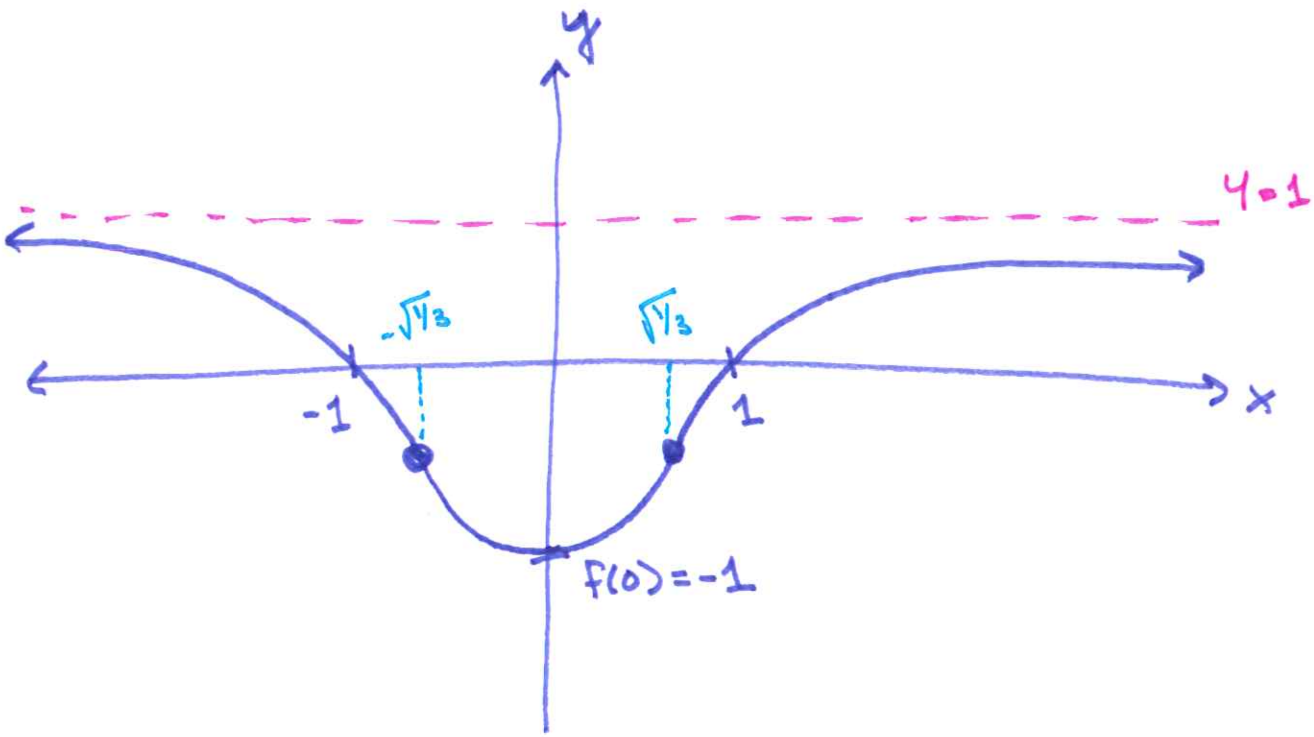
$$= \frac{(x^2+1)(4)(x^2+1-4x^2)}{(x^2+1)^4}$$

$$= \frac{-4 \cdot 3\left(x^2 - 1/3\right)}{(x^2+1)^4} = -12\,\frac{(x+1/\sqrt{3})(x-1/\sqrt{3})}{(x^2+1)^4}$$

so $f''(x) = 0$ when $\boxed{x = \pm 1/\sqrt{3}}$

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1 \qquad \lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1$$


1
1
f
+
0
−
0
+
−1
+1
f'
−
0
+
0
f''
−
+
−
$-\sqrt{1/3}$
$\sqrt{1/3}$


y
y = 1
$-\sqrt{1/3}$
$\sqrt{1/3}$
−1
1
x
f(0) = −1